

Fuzzy Extensions of OWL: Logical Properties and Reduction to Fuzzy Description Logics[☆]

G. Stoilos^{*,a,1}, G. Stamou^a, J. Z. Pan^b

^a*Department of Electrical and Computer Engineering,
National and Technical University of Athens*

^b*Department of Computing Science,
The University of Aberdeen*

Abstract

The Semantic Web is an extension of the current web, where information would have precisely defined meaning, based on formal semantics, and structured using a knowledge representational language. The current W3C standard for representing knowledge is the Web Ontology Language (OWL). OWL is based on Description Logics which is a popular knowledge representation formalism. Although, DLs are quite expressive they feature limitations with respect to what can be said about vague knowledge, which appears in several applications. Consequently, fuzzy extensions to OWL and DLs have gained considerable attention. In the current paper we study fuzzy extensions of the semantic web language OWL. First, we present the (abstract) syntax and semantics of a simplistic fuzzy extension of OWL creating fuzzy OWL (f-OWL). More importantly we use this extension to provide an investigation on the semantics of several f-OWL axioms and more precisely for those which can be expressed in many different ways (at least in classical DLs) and analyze their intuitive meaning. Moreover, we present a translation method which reduces inference problems of f-OWL into inference problems of expressive fuzzy Description Logics, in order to provide reasoning support through (fuzzy) DLs. Finally, we provide two further fuzzy extensions of OWL based on fuzzy subsumptions and fuzzy nominals.

Key words: Fuzzy OWL, Fuzzy Description Logics, Reduction of f-OWL, Syntactic Sugar Axioms, Reduction of f-Nominals.

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*Corresponding author

Email addresses: gstoil@image.ece.ntua.gr (G. Stoilos), gstam@softlab.ece.ntua.gr (G. Stamou), jpan@csd.abdn.ac.uk (J. Z. Pan)

¹Tel.: +30 210 7724352, Fax.: +30 210 7722492

1. Introduction

The last years great research effort has been focusing on the realization of the *Semantic Web* [1]. The Semantic Web has been proposed as an extension of the current web, where information and knowledge would be structured in a machine understandable and processable way. To this extent Semantic Web agents would be able to (semi)automatically carry out complex tasks assigned by humans in a meaningful (semantic) way. For example, they would be able to carry out a “holiday organization”, an “item purchase” or a “doctor appointment” [1] task. In order for information to be structured in a formal and machine understandable way the Semantic Web needs to make use of knowledge representational languages. The current W3C standard for representing knowledge in the Semantic Web is the OWL Web Ontology Language [2]. The logical underpinnings of OWL are mainly very expressive Description Logics, like *SHOIN(D)* and *SHIF(D)* [3]. Description Logics (DLs) is a logical formalism that has gained popularity the last decade. DLs combine both considerable expressive power as well as decidable reasoning procedures, which has attracted the attention of many researchers. Although, there is a close correspondence of OWL with Description Logics there are also many notable differences that need proper handling [4]. On the one hand in order for OWL to serve as a (Semantic) Web language it has adopted an XML like syntax to represent knowledge which is quite different from the usual abstract and logical syntax of knowledge representation languages. But, most importantly in order for OWL to be as much usable as possible and enjoy wide acceptance even by non-experts in knowledge representation it allows stating many complex DL axioms by simple *syntactic sugar* constructors which need to be properly mapped to DL axioms by reasoning systems. For example, one is able to declare that the range of role `hasSon` is the concept `Boy` by the statement `ObjectProperty(hasSon range(Boy))`, which in DLs would require a cumbersome axiom of the form $\top \sqsubseteq \forall \text{hasSon.Boy}$ or the equivalent $\exists \text{hasSon}^- . \top \sqsubseteq \text{Boy}$, where hasSon^- denotes the inverse of `hasSon`.

Although DLs is a quite expressive logical formalism it features limitations, mainly with what can be said about vague information. Such information is apparent in many applications and tasks both of the Semantic Web as well as of applications using DLs. For example, a task like a “holiday organization” could involve a request like: “Find me a *good* hotel in a place that is *relatively hot* and with *many* attractions”, or a “doctor appointment” could look like: “Make me an appointment with a doctor *close* to my home not *too early* and of *good* references”. Moreover, several intermediate processes, like information extraction or retrieval, matching user preferences with data and more, might involve imperfect information due to their automatic nature. Last but not least, several modern applications that have adopted Semantic Web technologies in order to enhance their performance and “connect” with the Semantic Web require the management of such knowledge. For example there are schemes for using Semantic Web technologies in multimedia applications for multimedia analysis [5], in Semantic Portals [6], ontology mapping validation [7],

Semantic Web Services matching [8], word-computing based systems [9] and many more, all of which require the management of some form of fuzzy information. For example, in image analysis one has to map low-level numerical values that are extracted by analysis algorithms for the color, shape and texture of a region into more high-level symbolic features like concepts. For example, values of the RGB color model would need to be mapped into concepts like `Blue`, `Green`, etc or values of special shape and texture (signal) transforms need to be mapped to concepts like `RectangularShaped`, `CoarseTextured`, `SmoothTextured` and more, all of which are obviously vague concepts and need proper handling.

In order to provide the necessary means to handle such information and knowledge there are today many proposals for fuzzy extensions to Description Logics [10, 11, 12] as well as several reasoning algorithms [13, 14, 15, 16, 17, 18] intended to provide inference support for vague information (see [19] for an overview). Fuzzy Description Logics extend the syntax and semantics of standard Description Logics using the ideas and techniques of Fuzzy Set Theory [20]. Consequently, instead of a Boolean $\{0, 1\}$ -interpretation of concepts and roles one adopts a more relaxed view where an object can belong to a set to any degree between 0 and 1. For example, a region *reg*₁ of an image could be `Blue` to a degree 0.6 and `closeTo` region *reg*₂ to a degree 0.8. Then we can use DL axioms which together with the above fuzzy assertions and reasoning mechanisms can be used to provide semantic means of object recognition [5]. For example, we can have the following axioms:

$$\begin{aligned} \text{Leafs} &\equiv \text{LightGreenColored} \sqcap \text{RegularTextured} \\ \text{Log} &\equiv \text{BrownColored} \sqcap \text{SmoothTextured} \\ \text{Tree} &\equiv \exists \text{hasPart} . (\text{Log} \sqcap \exists \text{isBelowOf} . \text{Leafs}) \end{aligned}$$

where \equiv is an equivalence relation, \sqcap is a conjunction, \exists is an existential restriction, `hasPart` and `isBelow` are roles (binary predicates), while the rest are concepts (unary predicates). Furthermore, in order to capture more accurately the semantic relations and properties of the entities of our domain, one might want to say that the role `hasPart` is transitive and that `isBelowOf` is the inverse of `isAboveOf`.

Although the literature on fuzzy extensions of DLs has been flourishing and we have also started to comprehend the difficulties of reasoning there are still several open issues regarding the semantics and their properties until we fully comprehend their logical features and provide proper ways to represent vague knowledge in the level of OWL. For example, as we have seen earlier there are at-least two different ways to map a role range axiom into a DL axiom. Although, these two different forms are logically equivalent in classical logics this is not always the case in fuzzy DLs. This is also true for many other syntactic sugar axioms, like concept disjointness, functional role axioms, domain and range restrictions and the one-of (enumeration) constructor. All these need to be clarified and investigated in order to further understand their logical properties and possibly provide guidelines for using a fuzzy OWL language. Last but not least, the reduction of several f-OWL reasoning services in expressive f-DL reasoning services has not been previously studied.

The current paper makes the following major contributions:

1. It presents fuzzy extensions of the OWL language. First, we present a simplistic extension that is only based on fuzzy instance relations and present its abstract syntax and semantics which are based on the notions of fuzzy sets and fuzzy set theoretic operators (see Section 4). To provide such semantics we mainly rely on the equivalence between OWL axioms and axioms of expressive fuzzy DLs. Finally we also present further extensions of OWL which allow for features such as fuzzy nominals (Section 6.1) and fuzzy subsumption (Section 6.2).
2. As we have already discussed there are usually more than one ways to map an OWL axiom into a DL axiom. The current paper provides a thorough investigation of these axioms intending to shed some light on their properties. More precisely, we investigate *class disjointness axioms*, *role range axioms*, *functional role axioms*, the *one-of/enumeration* constructor (Section 4.2) and the *fuzzy one-of/enumeration* constructor (Section 6.1). This is very important since although in classical logics the various different forms of these axioms and constructors are logically equivalent this is not true in fuzzy logics. Consequently, we compare the different semantics trying to explicate their intuitive meaning, assisting users and developers in choosing among them. Moreover, we also investigate in which special cases these different forms coincide. Finally, we also investigate on the semantics of fuzzy subsumptions. We believe that this analysis could assist (fuzzy) knowledge engineering tasks and users or system implementors to build tools for handling such knowledge.
3. For each of the above extensions, it shows how one can provide an RDF/XML serialization of the extended abstract syntax of fuzzy OWL. Using this syntax we can create real fuzzy OWL ontologies and store fuzzy information (Section 4.3).
4. It presents a translation method for reducing fuzzy OWL ontology entailment to fuzzy DL knowledge base satisfiability (Section 5). First, a method to map fuzzy OWL ontologies to fuzzy DL knowledge bases is provided, thus fuzzy OWL entailment is reduced to fuzzy DL knowledge base entailment. Subsequently, entailment of knowledge bases should be reduced to satisfiability, which is achieved by reducing every axiom of the knowledge base. Although the reduction of the most popular types of fuzzy DL axioms (e.g. concept subsumption and fuzzy concept assertions) to satisfiability has been studied in the literature there are plenty of axioms which have not. Thus, we additionally show how to reduce transitive role axioms, fuzzy role assertions and role subsumptions to the problem of satisfiability. Again, as in the case of syntactic sugar constructors there are cases where more than one reductions could be made according to which types of fuzzy operators are used. Our investigation is general enough to cover fuzzy DLs with arbitrary choices of fuzzy operators.
5. Besides showing the reduction to KB unsatisfiability we also investigate on what we call practical reductions. That is the reduction of several f-OWL axioms, like concept inclusions, require checking for an infinite number of unsatisfiable KBs. Straccia [13] has shown that for a specific class of fuzzy-DLs one can restrict to just two degrees.

We show that this result does not extend to fuzzy-DLs that use arbitrary fuzzy operators. Furthermore, for the case of fuzzy DLs considered in [13] we also show how this result can be extended for nominals (Theorem 5.6) and fuzzy nominals (Corollary 6.1), while we also show a practical reduction for fuzzy subsumption in f-DLs that use S -implications (Corollary 6.2).

At this point we want to make clear that our intention is not to present the ultimate or one and only fuzzy OWL extension. Throughout these years there have been several fuzzy features that have been proposed and added in Description Logics, like concept modifiers [12], fuzzy quantifiers [21], comparison expressions [22], and many more, that one could claim are missing from our presentation. Our goal here is to investigate the properties of fuzzy extensions to Semantic Web languages. More precisely, the syntactic sugar constructors, their meaning and finally the difficulties in reducing fuzzy OWL ontology entailment to fuzzy DL knowledge base satisfiability. A “standard” f-OWL language could only be the result of extensive face to face discussions between different research and industrial parties and identification of requirements in the context of a standardization group as well as evaluation of the efficiency and the existence of practical reasoning algorithms for each of those fuzzy features.

The rest of the paper is organized as follows. In Section 2 we provide a quick introduction to fuzzy set theory, expressive Description Logics and the OWL language. In Section 3 we present the syntax and semantics of a simplistic fuzzy extension of $SHOIN(\mathbf{D})$ creating the f- $SHOIN(\mathbf{D})$ DL. Although the semantics of f- $SHOIN(\mathbf{D})$ have been presented in the literature in the past [11, 23] we recall them here for completeness reasons. Subsequently, in Section 4 we present a (simplistic) fuzzy extension of OWL. We present the abstract syntax and provide semantics by relying on the equivalence between fuzzy OWL and expressive fuzzy Description Logics. Then, we use this simplistic extension as a mechanism for providing our investigation on the syntactic sugar axioms of f-OWL, like the disjoint classes, property range axioms and functional role axioms giving an insight on their meaning, properties and also investigating when different expressions of these constructors coincide. Having a mapping from f-OWL to f- $SHOIN(\mathbf{D})$, in Section 5 we finally present a technique that reduces the problem of f-OWL entailment to the problem of f- $SHOIN(\mathbf{D})$ knowledge base satisfiability, studying the reduction of the entailment of axioms that have not been studied before. In the same section we show that the result of Straccia [13] about reducing the number of degrees considered in the reduction of concept inclusion axioms cannot be extended in fuzzy DLs that use arbitrary fuzzy operators. Then, in Section 6 we present some further extension compared to the initial simplistic one. First, we extend f-OWL with the fuzzy one-of/enumeration constructor [24], while then we extend it with fuzzy subsumptions [11]. For both these reductions we present again the extended abstract syntax of f-OWL, its semantics, the reduction to satisfiability, and provide an investigation on these operators. Finally, Section 7 concludes the paper.

2. Preliminaries

2.1. Fuzzy Sets

Fuzzy set theory and fuzzy logic are widely used for capturing vague knowledge [20] in applications. While in classical set theory an element either belongs to a set or not, in fuzzy set theory elements belong only to a certain degree. More formally, let X be a collection of elements (the universe of discourse), i.e. $X = \{x_1, x_2, \dots\}$. A crisp subset S of X is any collection of elements of X that can be defined with the aid of its *characteristic function* $\chi_S(x)$ that assigns any $x \in X$ to the value 1 or 0 if this element belongs to X or not, respectively.

On the other hand, a fuzzy subset A of X , is defined by a *membership function* $\mu_A(x)$, or simply $A(x)$, $x \in X$, of the form $\mu_A(x) : X \rightarrow [0, 1]$. This membership function assigns any $x \in X$ to a value between 0 and 1 that represents the degree in which this element belongs to X . Similarly, we can define fuzzy relations. A fuzzy relation R over $X \times X$ is defined by a function which, given a pair of elements $\langle x, y \rangle$ returns the degree that the pair belongs to the fuzzy relation. Furthermore, the most important operations and properties defined on crisp sets and relations, like complement, union, intersection, transitivity etc, are extended in order to cover fuzzy sets and fuzzy relations, thus creating a sound mathematical theory which is today applied successfully in many applications.

Now we will introduce the fuzzy set theoretic operators.

The operation of a *fuzzy complement* (c) is a unary operation, defined by a function of the form $c : [0, 1] \rightarrow [0, 1]$. In order to produce meaningful fuzzy complements, these functions must satisfy certain properties. More precisely, they must satisfy the following conditions:

- *boundary conditions*: $c(0) = 1$ and $c(1) = 0$
- *monotonic decreasing*: for $a \leq b$, $c(a) \geq c(b)$.

Most of the cases fuzzy complements are also *continuous* and *involution*, for each $a \in [0, 1]$ $c(c(a)) = a$, holds. Many widely used fuzzy complements, like the Lukasiewicz negation, $c_L(a) = 1 - a$ and the Sugeno class, $c_S(a) = \frac{1-a}{1+\lambda a}$, $\lambda \in (-1, \infty)$ satisfy them. One non-*involution* fuzzy complement is the Gödel complement given by, $c_G(a) = 0$ if $a > 0$, otherwise $c(0) = 1$. The *equilibrium* of a fuzzy complement is defined as any point e_c for which $c(e_c) = e_c$.

The operation of *fuzzy intersection* is performed by a function of the form $t : [0, 1] \times [0, 1] \rightarrow [0, 1]$, called *t-norm* [20] operation. These functions must satisfy the following properties:

- *boundary condition*: $t(a, 1) = a$,
- *monotonic increasing*: for $b \leq d$, $t(a, b) \leq t(a, d)$

- *commutative*: $t(a, b) = t(b, a)$,
- *associative*: $t(a, t(b, c)) = t(t(a, b), c)$.

Usually, t-norm operations are also considered to be *continuous* and *subidempotent*, i.e. $t(a, a) < a$ for all $a \in (0, 1)$. Such norms are called *Archimedean* t-norms. The only *idempotent* t-norm is the Gödel t-norm given by, $t_G(a, b) = \min(a, b)$. It can be proved that for any t-norm t it holds that, $a, b \geq t(a, b)$, and $t(a, 0) = 0$. Commonly used Archimedean t-norms are the Lukasiewicz t-norm $t_L(a, b) = \max(0, a + b - 1)$, and the product t-norm $t_P(a, b) = a \cdot b$. A t-norm is called *nilpotent* if for every $a \in (0, 1]$ (called nilpotent element) there exists some $n \in \mathbb{N}$ such that:

$$t(\underbrace{a, a, \dots, a}_{n\text{-times}}) = 0.$$

For example, the Lukasiewicz t-norm is nilpotent since $t(0.3, 0.3) = \max(0, 0.3+0.3-1) = 0$.

The operation of *fuzzy union* is performed by a function $u : [0, 1] \times [0, 1] \rightarrow [0, 1]$, called *t-conorm*. Similarly to t-norms, these functions satisfy the boundary condition, $u(a, 0) = a$, are monotonic increasing, commutative and associative. In many cases t-conorms are continuous and *superidempotent*, $u(a, a) > a$ for all $a \in (0, 1)$. Such norms are called Archimedean t-conorms. The only idempotent t-conorm is the Gödel t-conorm given by, $u_G(a, b) = \max(a, b)$. It can be proved that for any t-conorm u it holds that, $a, b \leq u(a, b)$, and $u(a, 1) = 1$. Commonly used Archimedean t-conorms are the Lukasiewicz t-conorm $u_L(a, b) = \min(1, a + b)$, and the probabilistic sum $u_P(a, b) = a + b - a \cdot b$.

Another important operation in fuzzy logics is the *fuzzy implication*, which gives a truth value to the predicate $A \rightarrow B$. A fuzzy implication is a function \mathcal{J} of the form $\mathcal{J} : [0, 1] \times [0, 1] \rightarrow [0, 1]$, which is *monotonic decreasing (increasing)* on the first (second) argument. In fuzzy logics, we are usually interested in two kinds of fuzzy implications, i.e.,

- S-implications: $\mathcal{J}_S(a, b) = u(c(a), b)$,
- R-implications: $\mathcal{J}_R(a, b) = \sup\{x \in [0, 1] \mid t(a, x) \leq b\}$,

where a, b are the truth values for A and B , respectively. *S*-implications result from the extension of the proposition $\neg A \vee B$ with fuzzy operators, while *R*-implications by the proposition $\max\{x \in [0, 1] \mid a \wedge x \leq b\}$, which is an alternative expression for logical implication. Commonly used *R*-implications are the Lukasiewicz implication $\mathcal{J}_L(a, b) = \min(1, 1 - a + b)$, the Gödel implication, $\mathcal{J}_G(a, b) = b$, if $a > b$, $\mathcal{J}_G(a, b) = 1$ otherwise, and the Goguen implication, $\mathcal{J}_P(a, b) = a/b$, if $a > b$, $\mathcal{J}_P(a, b) = 1$ otherwise, while for *S*-implications the Kleene-Dienes implication, $\mathcal{J}_{KD}(a, b) = \max(1 - a, b)$. Note that the Lukasiewicz implication is both an *R* and an *S*-implication, i.e. $\sup\{x \mid t_L(a, x) \leq b\} = u_L(c_L(a), b)$. For each *R*-implication there is an associated fuzzy complement, called the *precomplement* of \mathcal{J} , defined by $c(a) = \mathcal{J}(a, 0)$. The precomplement is interesting when

one investigates meta-mathematical properties of fuzzy logics [25] but from a reasoning (practical) point of view is less interesting, since for both the Goguen and the Gödel R -implications their precomplement is the Gödel negation.

The above mentioned classes of fuzzy implications have some important differences. For example, for all R -implications $\mathcal{J}_R(a, b) = 1$ iff $a \leq b$, as well as for any R -implication \mathcal{J}_R , its respective t -norm t and $a, b, c \in [0, 1]$ it holds that $t(a, b) \leq c \Leftrightarrow \mathcal{J}_R(a, c) \geq b$. In other words \mathcal{J}_R and t are *adjoint* operators.

We conclude that in order to define a fuzzy logic we need to specify the fuzzy operations, c, t, u and \mathcal{J} , that we are going to use. Such a collection of operations would be referred to as a *fuzzy quadruple*, $\langle c, t, u, \mathcal{J} \rangle$, or *fuzzy triple* in the case of $\langle c, t, u \rangle$. In the current paper we will provide a general investigation of fuzzy DLs and fuzzy OWL, regardless of the norm operations used, while in some occasions we will go into more detail on the properties of fuzzy OWL and DLs when specific norm operations are used.

2.2. Expressive Description Logics

Description Logics (DLs) [3] are a family of logic-based knowledge representation formalisms designed to represent and reason about the knowledge of an application domain in a structured and well-understood way. They are based on a common family of languages, called *description languages*, which provide a set of constructors to build concept (class) and role (property) descriptions. Such descriptions can be used in axioms and assertions of DL knowledge bases and can be reasoned about with respect to (w.r.t.) DL knowledge bases by DL systems

In this section, we will briefly introduce the $SHOIN(\mathbf{D})$ DL, which will be extended later. A description language consists of an alphabet of distinct concept names (or atomic concepts) (\mathbf{C}), *abstract* role names (\mathbf{R}_A), *concrete* role names (\mathbf{R}_D)², abstract individuals (\mathbf{I}_A), concrete individuals (\mathbf{I}_D) and (concrete) datatypes (\mathbf{D})³. Subsequently, a set of constructors can be inductively applied over (atomic) concepts to define more complex ones.

The set of $SHOIN(\mathbf{D})$ -roles is defined by $\mathbf{R}_A \cup \{R^- \mid R \in \mathbf{R}_A\} \cup \mathbf{R}_D$, where R^- is called the *inverse role* of R . Let $A \in \mathbf{C}$, $R, S \in \mathbf{R}_A$ where S is a *simple role*⁴, $T_i \in \mathbf{R}_D$, $d \in \mathbf{D}$, $o \in \mathbf{I}_A$, $c \in \mathbf{I}_D$ and $n \in \mathbb{N}$, then $SHOIN(\mathbf{D})$ -concepts are defined inductively by the following production rule:

²Intuitively, abstract roles connect two (abstract) individuals/object, as e.g. `friends(george,tom)`, while concrete roles connect an individual with (concrete) individuals/datavalues, e.g. `hasAge(george, "28")`

³In First-Order Logic terminology, concepts are unary predicates ($C(x)$ where x is a variable), and roles are binary predicated ($R(x, y)$)

⁴A role is called simple if it is neither transitive nor it has any transitive sub-role. This restriction is crucial in order to retain decidability [26].

$$\begin{aligned}
C, D &\longrightarrow \perp \mid \top \mid A \mid C \sqcup D \mid C \sqcap D \mid \neg C \mid \forall R.C \mid \exists R.C \mid \geq pS \mid \leq pS \mid \\
&\quad \forall T.u \mid \exists T.u \mid \geq pT \mid \leq pT \mid \{o\} \\
u &\longrightarrow d \mid \{c\}
\end{aligned}$$

Concepts $\exists R.C$ and $\forall R.C$ are called existential and value restrictions, respectively. Concepts of the form $\leq nR$ and $\geq nR$ are called number restrictions, while concepts of the form $\{o\}$ nominals.

By restricting n to take only the values 0 and 1, i.e. concepts of the form $\leq 1R$, $\geq 1R$, $\geq 0R$ and $\leq 0R$, and by removing nominals (similarly with concrete individuals) we obtain the definition of $\mathcal{SHIF}(\mathbf{D})$ -concepts.

Description Logics have a model theoretic semantics, which are defined in terms of interpretations. An *interpretation* is a tuple $\mathcal{I} = (\Delta^{\mathcal{I}}, \Delta_{\mathbf{D}}, \cdot^{\mathcal{I}}, \cdot^{\mathbf{D}})$, where the abstract domain $\Delta^{\mathcal{I}}$ is a nonempty set of objects, the datatype domain $\Delta_{\mathbf{D}}$ is the domain of interpretation of all datatypes (disjoint from $\Delta^{\mathcal{I}}$) consisting of data values and $\cdot^{\mathcal{I}}$ and $\cdot^{\mathbf{D}}$ are two interpretation functions that map,

- each abstract individual $a \in \mathbf{I}_{\mathbf{A}}$ to an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$,
- each concrete individual $c \in \mathbf{I}_{\mathbf{D}}$ to an element $c^{\mathbf{D}} \in \Delta_{\mathbf{D}}$,
- each concept name $A \in \mathbf{C}$ to a subset $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$,
- each datatype d to a subset $d^{\mathbf{D}} \subseteq \Delta_{\mathbf{D}}$,
- each abstract role $R \in \mathbf{R}_{\mathbf{A}}$ to a relation $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ and
- each concrete role $T \in \mathbf{R}_{\mathbf{D}}$ to a relation $T^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta_{\mathbf{D}}$.

Interpretations can be extended to give semantics to arbitrary $\mathcal{SHOIN}(\mathbf{D})$ -concepts. These are depicted in Table 1, where $x, y \in \Delta^{\mathcal{I}}$ and $t \in \Delta_{\mathbf{D}}$.

A $\mathcal{SHOIN}(\mathbf{D})$ knowledge base (KB) consists of a TBox, an RBox and an ABox. A $\mathcal{SHOIN}(\mathbf{D})$ TBox is a finite set of *concept inclusion* (also called *subsumption*) axioms of the form $C \sqsubseteq D$, where C, D are $\mathcal{SHOIN}(\mathbf{D})$ -concepts. An interpretation \mathcal{I} satisfies $C \sqsubseteq D$ if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. Note that concept inclusion axioms of this form are called *General Concept Inclusions* (GCIs) [3], while if C is atomic, i.e. the subsumption is of the form $A \sqsubseteq D$, with $A \in \mathbf{C}$ we speak of *concept specifications*. If a TBox only includes concept specifications then it is called *simple*. Moreover, if a concept A is defined directly or indirectly with itself the TBox is called *cyclic*; in a different case it is called *acyclic*. A $\mathcal{SHOIN}(\mathbf{D})$ RBox is a finite set of *transitive role* axioms ($\text{Trans}(R)$), and *role inclusion* axioms ($R \sqsubseteq S$). An interpretation \mathcal{I} satisfies $\text{Trans}(R)$ if, for all $x, y, z \in \Delta^{\mathcal{I}}$, $\{\langle x, y \rangle, \langle y, z \rangle\} \subseteq R^{\mathcal{I}} \rightarrow \langle x, z \rangle \in R^{\mathcal{I}}$, and it satisfies $R \sqsubseteq S$ if $R^{\mathcal{I}} \subseteq S^{\mathcal{I}}$. A set of role inclusion axioms defines a *role hierarchy*. In some cases *functional role axioms* of the form $\text{Func}(R)$ are considered. An interpretation \mathcal{I} satisfies $\text{Func}(R)$ if $\forall b_1, b_2 \in \Delta^{\mathcal{I}}. R^{\mathcal{I}}(a, b_1) \wedge R^{\mathcal{I}}(a, b_2) \rightarrow b_1 = b_2$ [3]. As we will see in the following functional role axioms can be represented by regular concept

Table 1: Semantics of $SHOIN(\mathbf{D})$ -concepts

Constructor	Syntax	Semantics
top	\top	$\Delta^{\mathcal{I}}$
bottom	\perp	\emptyset
general negation	$\neg C$	$\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
conjunction	$C \sqcap D$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
disjunction	$C \sqcup D$	$C^{\mathcal{I}} \cup D^{\mathcal{I}}$
exists restriction	$\exists R.C$	$\{x \mid \exists y. \langle x, y \rangle \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\}$
value restriction	$\forall R.C$	$\{x \mid \forall y. \langle x, y \rangle \in R^{\mathcal{I}} \rightarrow y \in C^{\mathcal{I}}\}$
at-most restriction	$\leq pS$	$\{x \mid \#\{y \mid S^{\mathcal{I}}(x, y)\} \leq p\}$
at-least restriction	$\geq pS$	$\{x \mid \#\{y \mid S^{\mathcal{I}}(x, y)\} \geq p\}$
nominal	$\{o\}$	$\{o\}^{\mathcal{I}} = \{o^{\mathcal{I}}\}$
datatype exists	$\exists T.d$	$(\exists T.u)^{\mathcal{I}} = \{x \mid \exists t. \langle x, t \rangle \in T^{\mathcal{I}} \wedge t \in u^{\mathbf{D}}\}$
datatype value	$\forall T.d$	$(\forall T.u)^{\mathcal{I}} = \{x \mid \exists t. \langle x, t \rangle \in T^{\mathcal{I}} \rightarrow t \in u^{\mathbf{D}}\}$
datatype at-least	$\geq pT$	$(\geq pT)^{\mathcal{I}} = \{x \mid \#\{t \mid \langle x, t \rangle \in T^{\mathcal{I}}\} \geq p\}$
datatype at-most	$\leq pT$	$(\leq pT)^{\mathcal{I}} = \{x \mid \#\{t \mid \langle x, t \rangle \in T^{\mathcal{I}}\} \leq p\}$
datatype nominal	$\{c\}$	$\{c\}^{\mathcal{I}} = \{c^{\mathbf{D}}\}$

subsumption axioms. A $SHOIN$ *ABox* is a finite set of individual axioms (or assertions) of the form $a : C$, called *concept assertions*, or $(a, b) : R$, called *role assertions*, or $a \neq b$, stating that two individuals are different. An interpretation \mathcal{I} satisfies $a : C$ if $a^{\mathcal{I}} \in C^{\mathcal{I}}$, it satisfies $(a, b) : R$ if $\langle a^{\mathcal{I}}, b^{\mathcal{I}} \rangle \in R^{\mathcal{I}}$ and it satisfies $a \neq b$, if $a^{\mathcal{I}} \neq b^{\mathcal{I}}$.

2.3. The Web Ontology Language OWL

OWL is a standard (W3C recommendation) for expressing ontologies in the Semantic Web [2]. The OWL recommendation actually consists of three languages of increasing expressive power: OWL Lite, OWL DL and OWL Full. *OWL Lite* and *OWL DL* are basically very expressive Description Logics (DLs); they are almost⁵ equivalent to the $SHIF(\mathbf{D})$ and $SHOIN(\mathbf{D})$ DLs [2]. Furthermore, there are some syntactic differences, e.g. OWL has an RDF/XML syntax as well as an abstract syntax that is slightly different than that of DLs.. Furthermore, there are several syntactic sugar axioms for encapsulating and hiding complex DL axioms. *OWL Full* provides the same set of constructors as OWL DL, but allows them to be used in an unconstrained way, creating meta-modelling statements [27]. Because of this feature OWL Full has been proved to be undecidable [27]; therefore, when we mention OWL in this paper, we usually mean OWL DL.

Let \mathbf{C} , \mathbf{R}_A , \mathbf{R}_D , \mathbf{I}_A and \mathbf{I}_D be the sets of class names, object property names, datatype property names, abstract individuals and concrete individuals, respectively. Note that in OWL terminology DL roles are just called properties. An OWL DL *interpretation* is fairly

⁵They also provide annotation properties, which Description Logics do not.

Table 2: OWL Class and Property Descriptions

Abstract Syntax	DL Syntax	Semantics
Class(A)	A	$A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$
owl:Thing	\top	$\top^{\mathcal{I}} = \Delta^{\mathcal{I}}$
owl:Nothing	\perp	$\perp^{\mathcal{I}} = \emptyset$
intersectionOf($C_1 C_2 \dots$)	$C_1 \sqcap C_2$	$(C_1 \sqcap C_2)^{\mathcal{I}} = C_1^{\mathcal{I}} \cap C_2^{\mathcal{I}}$
unionOf($C_1 C_2 \dots$)	$C_1 \sqcup C_2$	$(C_1 \sqcup C_2)^{\mathcal{I}} = C_1^{\mathcal{I}} \cup C_2^{\mathcal{I}}$
complementOf(C)	$\neg C$	$(\neg C)^{\mathcal{I}} = \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
oneOf($o_1 o_2 \dots$)	$\{o_1\} \sqcup \{o_2\}$	$(\{o_1\} \sqcup \{o_2\})^{\mathcal{I}} = \{o_1^{\mathcal{I}}, o_2^{\mathcal{I}}\}$
restriction(R someValuesFrom(C))	$\exists R.C$	$(\exists R.C)^{\mathcal{I}} = \{x \mid \exists y. \langle x, y \rangle \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\}$
restriction(R allValuesFrom(C))	$\forall R.C$	$(\forall R.C)^{\mathcal{I}} = \{x \mid \forall y. \langle x, y \rangle \in R^{\mathcal{I}} \rightarrow y \in C^{\mathcal{I}}\}$
restriction(R value(o))	$\exists R.\{o\}$	$(\exists R.\{o\})^{\mathcal{I}} = \{x \mid \langle x, o^{\mathcal{I}} \rangle \in R^{\mathcal{I}}\}$
restriction(S minCardinality(p))	$\geq pS$	$(\geq pS)^{\mathcal{I}} = \{x \mid \#\{y. \langle x, y \rangle \in S^{\mathcal{I}}\} \geq p\}$
restriction(S maxCardinality(p))	$\leq pS$	$(\leq pS)^{\mathcal{I}} = \{x \mid \#\{y. \langle x, y \rangle \in S^{\mathcal{I}}\} \leq p\}$
oneOf($c_1 c_2 \dots$)	$\{c_1\} \sqcup \{c_2\}$	$(\{c_1\} \sqcup \{c_2\})^{\mathcal{I}} = \{c_1^{\mathcal{D}}, c_2^{\mathcal{D}}\}$
restriction(T someValuesFrom(u))	$\exists T.u$	$(\exists T.u)^{\mathcal{I}} = \{x \mid \exists t. \langle x, t \rangle \in T^{\mathcal{I}} \wedge t \in u^{\mathcal{D}}\}$
restriction(T allValuesFrom(u))	$\forall T.u$	$(\forall T.u)^{\mathcal{I}} = \{x \mid \forall t. \langle x, t \rangle \in T^{\mathcal{I}} \rightarrow t \in u^{\mathcal{D}}\}$
restriction(T value(c))	$\exists T.\{c\}$	$(\exists T.\{c\})^{\mathcal{I}} = \{x \mid \langle x, c^{\mathcal{D}} \rangle \in T^{\mathcal{I}}\}$
restriction(T minCardinality(p))	$\geq pT$	$(\geq pT)^{\mathcal{I}} = \{x \mid \#\{t \mid \langle x, t \rangle \in T^{\mathcal{I}}\} \geq p\}$
restriction(T maxCardinality(p))	$\leq pT$	$(\leq pT)^{\mathcal{I}} = \{x \mid \#\{t \mid \langle x, t \rangle \in T^{\mathcal{I}}\} \leq p\}$
ObjectProperty(S)	S	$S^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$
ObjectProperty(S' inverseOf(S))	S^-	$(S^-)^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$
DatatypeProperty(T)	T	$T^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta_{\mathcal{D}}$

standard by Description Logic standards. Thus, again we have a tuple $\mathcal{I} = (\Delta^{\mathcal{I}}, \Delta_{\mathcal{D}}, \cdot^{\mathcal{I}}, \cdot^{\mathcal{D}})$, where the abstract domain $\Delta^{\mathcal{I}}$ is a nonempty set of objects, the datatype domain $\Delta_{\mathcal{D}}$ is the domain of interpretation of all datatypes (disjoint from $\Delta^{\mathcal{I}}$) consisting of data values and $\cdot^{\mathcal{I}}$ and $\cdot^{\mathcal{D}}$ are two interpretation functions that map, abstract individual and concrete individuals as before, class names as concept names, datatypes as before, object properties as abstract roles and datatype properties as concrete roles. Then it can be extended to complex OWL class and property descriptions. Table 2 presents the abstract syntax of OWL class and property descriptions, their corresponding DL syntax and finally the semantics of these descriptions which is an immediate result of the mapping to DL concepts and roles.

There are some remarks regarding Table 2. First, we can see that the *one-of* constructor [3] (with DL syntax $\{o_1, \dots, o_m\}$) allowed in OWL is a syntactic sugar in the presence of nominals and disjunction, i.e. $\{o_1, \dots, o_m\} \equiv \{o_1\} \sqcup \dots \sqcup \{o_m\}$. Since such constructors are allowed in OWL, such axioms are directly mapped to them. Similarly, the *value* operator (called *fills* constructor in DLs [3] with syntax $R : o$), is a syntactic sugar in the presence of nominals and existential restrictions. More precisely, $R : o \equiv \exists R.\{o\}$.

Subsequently, OWL allows class and property axioms. As with class and property descriptions, OWL axioms can be seen as DL axioms. Even axioms that seem to provide more expressive power to OWL can still be mapped to DL axioms. Table 3 presents the

Table 3: OWL Class and Property Axioms

Abstract Syntax	DL Syntax	Semantics
Class(A partial $C_1 \dots C_n$) Class(A complete $C_1 \dots C_n$) EnumeratedClass(A $o_1 \dots o_n$) SubClassOf(C_1, C_2) EquivalentClasses($C_1 \dots C_n$) DisjointClasses($C_1 \dots C_n$)	$A \sqsubseteq C_1 \sqcap \dots \sqcap C_n$ $A \equiv C_1 \sqcap \dots \sqcap C_n$ $A \equiv \{o_1\} \sqcup \dots \sqcup \{o_n\}$ $C_1 \sqsubseteq C_2$ $C_1 \equiv \dots \equiv C_n$ $C_i \sqsubseteq \neg C_j$	$A^{\mathcal{I}} \subseteq C_1^{\mathcal{I}} \cap \dots \cap C_n^{\mathcal{I}}$ $A^{\mathcal{I}} = C_1^{\mathcal{I}} \cap \dots \cap C_n^{\mathcal{I}}$ $A^{\mathcal{I}} = \{o_1^{\mathcal{I}}, \dots, o_n^{\mathcal{I}}\}$ $C_1^{\mathcal{I}} \subseteq C_2^{\mathcal{I}}$ $C_1^{\mathcal{I}} = \dots = C_n^{\mathcal{I}}$ $C_i^{\mathcal{I}} \subseteq (\neg C_j)^{\mathcal{I}}, 1 \leq i < j \leq n$
SubPropertyOf(R_1, R_2) EquivalentProperties($R_1 \dots R_n$) ObjectProperty(R super(R_1) ... super(R_n) domain(C_1) ... domain(C_k) range(C_1) ... range(C_h) [InverseOf(S)] [Symmetric] [Functional] [InverseFunctional] [Transitive])	$R_1 \sqsubseteq R_2$ $R_1 \equiv \dots \equiv R_n$ $R \sqsubseteq R_i$ $\exists R. \top \sqsubseteq C_i$ $\top \sqsubseteq \forall R. C_i$ $R \equiv S^{-}$ $R \equiv R^{-}$ $\top \sqsubseteq \leq 1R$ $\top \sqsubseteq \leq 1R^{-}$ Trans (R)	$R_1^{\mathcal{I}} \subseteq R_2^{\mathcal{I}}$ $R_1^{\mathcal{I}} = \dots = R_n^{\mathcal{I}}$ $R^{\mathcal{I}} \subseteq R_i^{\mathcal{I}}$ $R^{\mathcal{I}} \subseteq C_1^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times C_i^{\mathcal{I}}$ $R^{\mathcal{I}} = (S^{-})^{\mathcal{I}}$ $R^{\mathcal{I}} = (R^{-})^{\mathcal{I}}$ $\forall x \in \Delta^{\mathcal{I}}. \#\{y \mid \langle x, y \rangle \in R^{\mathcal{I}}\} \leq 1$ $\forall x \in \Delta^{\mathcal{I}}. \#\{y \mid \langle x, y \rangle \in (R^{-})^{\mathcal{I}}\} \leq 1$ $\{\langle x, y \rangle, \langle y, z \rangle\} \subseteq R^{\mathcal{I}} \rightarrow \langle x, z \rangle \in R^{\mathcal{I}}$
SubPropertyOf(T_1, T_2) EquivalentProperties($R_1 \dots R_n$) ObjectProperty(T super(T_1) ... super(T_n) domain(C_1) ... domain(C_k) range(d_1) ... range(d_h) [Functional])	$T_1 \sqsubseteq T_2$ $T_1 \equiv \dots \equiv T_n$ $T \sqsubseteq T_i$ $\exists T. \top \sqsubseteq C_i$ $\top \sqsubseteq \forall T. d_i$ $\top \sqsubseteq \leq 1T$	$T_1^{\mathcal{I}} \subseteq T_2^{\mathcal{I}}$ $T_1^{\mathcal{I}} = \dots = T_n^{\mathcal{I}}$ $T^{\mathcal{I}} \subseteq T_i^{\mathcal{I}}$ $T^{\mathcal{I}} \subseteq C_1^{\mathcal{I}} \times \Delta_{\mathbf{D}}^{\mathcal{I}}$ $T^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times d_i^{\mathbf{D}}$ $\forall x \in \Delta^{\mathcal{I}}. \#\{t \mid \langle x, t \rangle \in T^{\mathcal{I}}\} \leq 1$
Individual(o type(C_1) ... type(C_n) value(R_1, o_1) ... value(R_n, o_n)) Sameindividual($o_1 \dots o_n$) DifferentIndividuals($o_1 \dots o_n$)	$o : C_i$ $(o, o_i) : R_i$ $o_1 = \dots = o_n$ $o_i \neq o_j$	$o^{\mathcal{I}} \in C_i^{\mathcal{I}}, 1 \leq i \leq n$ $\langle o^{\mathcal{I}}, o_i^{\mathcal{I}} \rangle \in R_i^{\mathcal{I}}, 1 \leq i \leq n$ $o_1^{\mathcal{I}} = \dots = o_n^{\mathcal{I}}$ $o_i^{\mathcal{I}} \neq o_j^{\mathcal{I}}, 1 \leq i < j \leq n$

abstract syntax of OWL class and property axioms. It also presents the corresponding DL axiom which in turn gives rise to the semantics of the axiom. Again as one can note from Table 3 enumerated classes, property domain and range axioms, class disjointness axioms and functional role axioms are just syntactic sugar and can be represented using regular, but sometimes cumbersome, DL axioms.

3. A Simplistic Fuzzy Extension of the $\mathcal{SHOIN}(\mathbf{D})$ DL

In this section we present a (simplistic) fuzzy extension of the f- $\mathcal{SHOIN}(\mathbf{D})$ DL. Up to now several fuzzy extensions of DL languages have been presented in the literature, advocating for the need for different “fuzzy” features which have been traditionally investigated and proposed in the Fuzzy Set literature [20]. For example, Straccia proposed the fuzzification of concept inclusions [11], Bobillo et al. proposed a fuzzy extension of the nominal constructor creating fuzzy nominals [24], Sánchez and Tettamanzi [21] proposed the use of fuzzy quantifiers, Hölldobler et al. [28] proposed fuzzy concept modifiers, Kang et al. proposed comparison expressions [22]. We will not attempt to present and investigate all these features, thus we are rather going to start with a simplistic extension based only

on fuzzy assertions (like the one presented in [29]) which is enough to present our main results, while in subsequent section we will investigate some of these features.

As usual we have an alphabet of distinct concept names (\mathbf{C}), abstract role names (\mathbf{R}_A), concrete role names (\mathbf{R}_D), abstract individuals (\mathbf{I}_A) and concrete individuals (\mathbf{I}_D). The set of $\mathit{SHOIN}(\mathbf{D})$ -roles is defined by $\mathbf{R}_A \cup \{R^- \mid R \in \mathbf{R}_A\} \cup \mathbf{R}_D$, where R^- is called the *inverse role* of R . Let $A \in \mathbf{C}$, $R, S \in \mathbf{R}_A$ where S is a *simple role*⁶, $T_i \in \mathbf{R}_D$, d is a datatype, $o, o_1, \dots, o_k \in \mathbf{I}_A$, $c, n \in (0, 1]$ and $p, k \in \mathbb{N}$, then $f\text{-}\mathit{SHOIN}(\mathbf{D})$ -concepts are defined inductively by the following production rule:

$$\begin{aligned} C, D &\longrightarrow \perp \mid \top \mid A \mid C \sqcup D \mid C \sqcap D \mid \neg C \mid \forall R.C \mid \exists R.C \mid \geq pS \mid \leq pS \mid \\ &\quad \forall T.u \mid \exists T.u \mid \geq pT \mid \leq pT \mid \{o\} \mid \{o_1, \dots, o_k\} \mid R : o \\ u &\longrightarrow d \mid \{c\} \end{aligned}$$

As we can see $f\text{-}\mathit{SHOIN}(\mathbf{D})$ -concepts of our simplistic extension are formed by a similar abstract syntax as that of crisp $\mathit{SHOIN}(\mathbf{D})$ -concepts.

The semantics of fuzzy DLs are provided by a *fuzzy interpretation* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ [13] together with an interpretation of the datatype (concrete) domain $\mathbf{D} = (\Delta_{\mathbf{D}}, \cdot^{\mathbf{D}})$ [11]. Hence, a fuzzy interpretation is defined by a 4-tuple $\mathcal{I} = (\Delta^{\mathcal{I}}, \Delta_{\mathbf{D}}, \cdot^{\mathcal{I}}, \cdot^{\mathbf{D}})$, where the abstract domain $\Delta^{\mathcal{I}}$ is a non-empty set of objects, the datatype domain $\Delta_{\mathbf{D}}$ is the domain of interpretation of all datatypes (disjoint from $\Delta^{\mathcal{I}}$) consisting of data values, and $\cdot^{\mathcal{I}}$ and $\cdot^{\mathbf{D}}$ are two fuzzy interpretation functions, which map

- an abstract individual a to an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$,
- a concrete individual c to an element $c^{\mathbf{D}} \in \Delta_{\mathbf{D}}$,
- a concept name A to a function $A^{\mathcal{I}} : \Delta^{\mathcal{I}} \rightarrow [0, 1]$,
- an abstract role name R to a function $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \rightarrow [0, 1]$,
- a datatype d to a function $d^{\mathbf{D}} : \Delta_{\mathbf{D}} \rightarrow [0, 1]$, and
- a concrete role name T to a function $T^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta_{\mathbf{D}} \rightarrow [0, 1]$.

As we see we have used the concept of fuzzy datatypes introduced in [11]. Intuitively, an object (pair of objects) can now belong to a fuzzy concept (role) to any degree between 0 and 1. For example, $\text{HotPlace}^{\mathcal{I}}(\text{Rome}^{\mathcal{I}}) = 0.7$, means that $\text{Rome}^{\mathcal{I}}$ is a hot place to a degree equal to 0.7. Moreover, fuzzy interpretations can be extended to interpret $f\text{-}\mathit{SHOIN}(\mathbf{D})$ -concepts and roles, with the aid of the fuzzy set theoretic operations, defined in Section 2.1. The complete semantics are depicted in Table 4.

⁶A role is called simple if it is neither transitive nor it has any transitive sub-role. This restriction is crucial in order to retain decidability [26]

Table 4: Syntax and Semantics of f-*SHOIN*(**D**)-concepts

Constructor	DL Syntax	Semantics
top concept	\top	$\top^{\mathcal{I}}(a) = 1$
bottom	\perp	$\perp^{\mathcal{I}}(a) = 0$
data value	c	$c^{\mathcal{I}} = c^{\mathbf{D}}$
datatype	d	$d^{\mathcal{I}}(y) = d^{\mathbf{D}}(y)$
conjunction	$C \sqcap D$	$(C \sqcap D)^{\mathcal{I}}(a) = t(C^{\mathcal{I}}(a), D^{\mathcal{I}}(a))$
disjunction	$C \sqcup D$	$(C \sqcup D)^{\mathcal{I}}(a) = u(C^{\mathcal{I}}(a), D^{\mathcal{I}}(a))$
negation	$\neg C$	$(\neg C)^{\mathcal{I}}(a) = c(C^{\mathcal{I}}(a))$
nominal	$\{o\}$	$\{o\}^{\mathcal{I}}(a) = 1$ iff $o^{\mathcal{I}} = a$, $\{o\}^{\mathcal{I}}(a) = o$ otherwise
one-of	$\{o_1, \dots, o_k\}$	$\{o_1, \dots, o_k\}^{\mathcal{I}}(a) = 1$ if $a \in \{o_1^{\mathcal{I}}, \dots, o_k^{\mathcal{I}}\}$, 0 otherwise
fills	$R : o$	$(R : o)^{\mathcal{I}}(a) = R^{\mathcal{I}}(a, o^{\mathcal{I}})$
existential restriction	$\exists R.C$	$(\exists R.C)^{\mathcal{I}}(a) = \sup_{b \in \Delta^{\mathcal{I}}} t(R^{\mathcal{I}}(a, b), C^{\mathcal{I}}(b))$
value restriction	$\forall R.C$	$(\forall R.C)^{\mathcal{I}}(a) = \inf_{b \in \Delta^{\mathcal{I}}} \mathcal{J}(R^{\mathcal{I}}(a, b), C^{\mathcal{I}}(b))$
at-least restriction	$\geq pS$	$(\geq pS)^{\mathcal{I}}(a) = \sup_{b_1, \dots, b_p \in \Delta^{\mathcal{I}}} t(\prod_{i=1}^p S^{\mathcal{I}}(a, b_i), \prod_{i < j} \{b_i \neq b_j\})$
at-most restriction	$\leq pS$	$(\leq pS)^{\mathcal{I}}(a) = \inf_{b_1, \dots, b_{p+1} \in \Delta^{\mathcal{I}}} \mathcal{J}(\prod_{i=1}^{p+1} S^{\mathcal{I}}(a, b_i), \prod_{i < j} \{b_i = b_j\})$
inverse role	R^-	$(R^-)^{\mathcal{I}}(b, a) = R^{\mathcal{I}}(a, b)$
datatype exists	$\exists T.d$	$(\exists T.d)^{\mathcal{I}}(a) = \sup_{y \in \Delta_{\mathbf{D}}} t(T^{\mathcal{I}}(a, y), d^{\mathcal{I}}(y))$
datatype value	$\forall T.d$	$(\forall T.d)^{\mathcal{I}}(a) = \inf_{y \in \Delta_{\mathbf{D}}} \mathcal{J}(T^{\mathcal{I}}(a, y), d^{\mathcal{I}}(y))$
datatype at-least	$\geq pT$	$(\geq pT)^{\mathcal{I}}(a) = \sup_{y_1, \dots, y_p \in \Delta_{\mathbf{D}}} t(\prod_{i=1}^p R^{\mathcal{I}}(a, y_i), \prod_{i < j} \{y_i \neq y_j\})$
datatype at-most	$\leq pT$	$(\leq pT)^{\mathcal{I}}(a) = \inf_{y_1, \dots, y_{p+1} \in \Delta_{\mathbf{D}}} \mathcal{J}(\prod_{i=1}^{p+1} R^{\mathcal{I}}(a, y_i), \prod_{i < j} \{y_i = y_j\})$
datatype nominal	$\{c\}$	$\{c\}^{\mathcal{I}}(y) = 1$ iff $c^{\mathbf{D}} = y$, $\{c\}^{\mathbf{D}}(y) = o$ otherwise

There are some remarks regarding Table 4. Although, there are several proposals for providing semantics for number restrictions, some examples are [21, 11, 30, 31, 32], we have chosen to follow the semantics proposed in [11], later revised in [23]. That is because, as showed in [14], under these semantics there exist efficient procedures for deciding the key inference problems of fuzzy DLs, like entailment and subsumption. The key property here is that the equalities ($=$) and inequalities (\neq) of objects of $\Delta^{\mathcal{I}}$ and $\Delta_{\mathbf{D}}$ in the semantic function of number restrictions are considered as crisp. In a case the *similarity* measure between objects should have been considered, which we do not know how to handle in reasoning tasks such as entailment. Note also that recently there was a further proposal for refining these semantics [33]. More precisely, in the semantic function of number restrictions, the authors have replaced the t -norm product $\prod_{i=1}^p R^{\mathcal{I}}(a, y_i)$ with \min . The intuition again is to retain the counting property that provides nice reasoning properties [14].

A fuzzy TBox is a finite set of fuzzy concept axioms. Let C and D be f-*SHOIN*(**D**)-concepts. Fuzzy concept axioms of the form $C \sqsubseteq D$ are called *fuzzy concept inclusion axioms*, while fuzzy concept axioms of the form $C \equiv D$ are called *fuzzy equivalence axioms*.

A fuzzy interpretation \mathcal{I} satisfies $C \sqsubseteq D$ if $\forall a \in \Delta^{\mathcal{I}}, C^{\mathcal{I}}(a) \leq D^{\mathcal{I}}(a)$ and it satisfies $C \equiv D$ if $C^{\mathcal{I}}(a) = D^{\mathcal{I}}(a)$. Finally, a fuzzy interpretation \mathcal{I} satisfies an f-*SHOIN*(\mathbf{D}) TBox \mathcal{T} if it satisfies each axiom in \mathcal{T} ; then we say that \mathcal{I} is a *model* of \mathcal{T} . Please note that in this simplistic extension we give a crisp subsumption of fuzzy concepts. These semantics differ from the ones in [11], where a *fuzzy subsumption* of fuzzy concepts was provided, investigated in Section 6.2.

A fuzzy RBox is a finite set of fuzzy role axioms. Let $R, S \in \mathbf{R}_{\mathbf{A}}$ and $T, U \in \mathbf{R}_{\mathbf{D}}$. Then fuzzy role axioms of the form $\text{Trans}(R)$, are called *fuzzy transitive role* axioms, while fuzzy role axioms of the form $R \sqsubseteq S$ or $T \sqsubseteq U$ are called *fuzzy role inclusion* axioms. A fuzzy interpretation \mathcal{I} satisfies $\text{Trans}(R)$ if $\forall a, c \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(a, c) \geq \sup_{b \in \Delta^{\mathcal{I}}} \{t(R^{\mathcal{I}}(a, b), R^{\mathcal{I}}(b, c))\}$, it satisfies $R \sqsubseteq S$ if $\forall \langle a, b \rangle \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}, R^{\mathcal{I}}(a, b) \leq S^{\mathcal{I}}(a, b)$, and it satisfies $T \sqsubseteq U$ if $\forall \langle a, y \rangle \in \Delta^{\mathcal{I}} \times \Delta_{\mathbf{D}}, T^{\mathcal{I}}(a, y) \leq U^{\mathcal{I}}(a, y)$. Finally, \mathcal{I} satisfies an f-*SHOIN*(\mathbf{D}) RBox if it satisfies each role axiom in \mathcal{R} ; in this case we say that \mathcal{I} is a model of \mathcal{R} .

A fuzzy ABox is a finite set of fuzzy assertions. A *fuzzy assertion* [13] is of the form $(a : C) \bowtie n, ((a, b) : R) \bowtie n$, where $\bowtie \in \{=, \geq, >, \leq, <\}$, $a \doteq b$ or $a \neq b$, for $a, b \in \mathbf{I}_{\mathbf{A}}$. For a fuzzy interpretation \mathcal{I} ,

$$\begin{aligned} \mathcal{I} \text{ satisfies } (a : C) \geq n & \text{ if } C^{\mathcal{I}}(a^{\mathcal{I}}) \geq n, \\ \mathcal{I} \text{ satisfies } ((a, b) : R) \geq n & \text{ if } R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq n, \\ \mathcal{I} \text{ satisfies } (a : C) \leq n & \text{ if } C^{\mathcal{I}}(a^{\mathcal{I}}) \leq n, \\ \mathcal{I} \text{ satisfies } ((a, b) : R) \leq n & \text{ if } R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \leq n, \\ \mathcal{I} \text{ satisfies } a \doteq b & \text{ if } a^{\mathcal{I}} = b^{\mathcal{I}}, \\ \mathcal{I} \text{ satisfies } a \neq b & \text{ if } a^{\mathcal{I}} \neq b^{\mathcal{I}}. \end{aligned}$$

The satisfiability of fuzzy assertions with $>$ and $<$ is defined similarly. A fuzzy interpretation \mathcal{I} satisfies a fuzzy ABox \mathcal{A} if it satisfies all fuzzy assertions in \mathcal{A} . In this case, we say that \mathcal{I} is a *model* of \mathcal{A} . If \mathcal{A} has a model then we say that it is *consistent*, otherwise it is *inconsistent*.

A *fuzzy knowledge base* Σ is a triple $\langle \mathcal{T}, \mathcal{R}, \mathcal{A} \rangle$, that contains a fuzzy TBox, RBox and ABox, respectively. A fuzzy interpretation \mathcal{I} satisfies an f-*SHOIN*(\mathbf{D}) knowledge base Σ if it satisfies all axioms in Σ ; in this case, \mathcal{I} is called a *model* of Σ .

We conclude this section by showing that fuzzy DLs are a *conservative* extension of classical DLs in the sense that if we restrict to the truth set of Boolean logic, i.e. to the values zero and one, we obtain the classical DLs.

Theorem 3.1. *Fuzzy interpretations coincide with crisp interpretations if we restrict to the membership degrees of 0 and 1.*

Proof: The proof is given in the appendix ■

3.1. Inference Problems of Fuzzy DLs

Similarly to classical DLs, f-DLs also offer a set of inference services. Below, we summarize the most important inference services of fuzzy DLs.

- **KB Satisfiability:** An f-*SHOIN*(**D**) knowledge base $\Sigma = \langle \mathcal{T}, \mathcal{R}, \mathcal{A} \rangle$ is *satisfiable* (unsatisfiable) iff there exists (does not exist) a fuzzy interpretation \mathcal{I} which satisfies all axioms in Σ .
- **Concepts n -satisfiability:** An f-*SHOIN*(**D**)-concept C is *n -satisfiable* w.r.t. Σ iff there exists a model \mathcal{I} of Σ in which there exists some $a \in \Delta^{\mathcal{I}}$ such that $C^{\mathcal{I}}(a) = n$, and $n \in (0, 1]$.
- **Concept Subsumption:** A fuzzy concept C is subsumed by D w.r.t. Σ iff in every model \mathcal{I} of Σ we have $\forall d \in \Delta^{\mathcal{I}}, C^{\mathcal{I}}(d) \leq D^{\mathcal{I}}(d)$.
- **ABox Consistency:** An f-*SHOIN*(**D**) ABox \mathcal{A} is *consistent* (*inconsistent*) w.r.t. a TBox \mathcal{T} and an RBox \mathcal{R} if there exists (does not exist) a model \mathcal{I} of \mathcal{T} and \mathcal{R} which satisfies every assertion in \mathcal{A} .
- **Entailment:** Given an axiom Ψ , we say that Σ *entails* Ψ , writing $\Sigma \models \Psi$, iff every model \mathcal{I} of Σ satisfies Ψ .
- **Greater Lower Bound (glb):** The *greatest lower bound* of an assertion Φ w.r.t. Σ is defined as,

$$glb(\Sigma, \Phi) = \sup\{n \mid \Sigma \models \Phi \geq n\}, \text{ where } \sup \emptyset = 0.$$

As it has been shown in the literature, all of the above inference problems of fuzzy DLs w.r.t. a knowledge base Σ can be reduced to knowledge base satisfiability [13]. More precisely, let $\Sigma = \langle \mathcal{T}, \mathcal{R}, \mathcal{A} \rangle$ be an f-*SHOIN*(**D**) KB. Then we have the following equivalences:

$$\begin{aligned} C \text{ is } n\text{-satisfiable w.r.t. } \Sigma & \quad \text{iff} \quad \langle \mathcal{T}, \mathcal{R}, \mathcal{A} \cup \{(a : C) \geq n\} \rangle \text{ is satisfiable} \\ C \sqsubseteq D \text{ w.r.t. } \Sigma & \quad \text{iff} \quad \langle \mathcal{T}, \mathcal{R}, \mathcal{A} \cup \{(a : C) \geq n, (a : D) < n\} \rangle \\ & \quad \text{is unsatisfiable for every } n \in [0, 1] \\ \Sigma \models (a : C) \triangleright n & \quad \text{iff} \quad \langle \mathcal{T}, \mathcal{R}, \mathcal{A} \cup \{(a : C) \neg \triangleright n\} \rangle \text{ is unsatisfiable.} \end{aligned}$$

There are some remarks regarding the above reductions. Firstly, note that traditionally in DLs only the reduction of the entailment of concept axioms and (fuzzy) assertions is considered. Differently, since in the following we will need to reduce the entailment of knowledge bases (i.e. of arbitrary axioms) to KB unsatisfiability, in Section 5 we will extend these results further. Secondly, note that the subsumption problem requires checking for unsatisfiability for every degree $n \in [0, 1]$. Obviously, this is practically impossible. Straccia proves [13] that for a specific class of fuzzy DLs, which we call f_{KD} -DLs (see below for a definition) it suffices to check for the unsatisfiability of the knowledge base only for two randomly selected values each one from the intervals $(0, 0.5]$ and $(0.5, 1]$. More precisely, $C \sqsubseteq D$ w.r.t. Σ iff the knowledge base $\langle \mathcal{T}, \mathcal{R}, \mathcal{A} \cup \{(a : C) \geq n, (a : D) < n\} \rangle$ is unsatisfiable

for each $n \in \{n_1, n_2\}$, where $n_1 \in (0, 0.5]$ and $n_2 \in (0.5, 1]$. In Section 5 we will show that this result *does not* generalize to fuzzy DLs that use arbitrary fuzzy operators, while it generalizes to f_{KD} -DLs that use any arbitrary continuous complement. Still in the case that nominals are allowed (i.e. $f_{KD}\text{-}\mathcal{SHOIN}(\mathbf{D})$) more care should be paid. In the following we refer to such reductions as *practical reductions*.

Example 3.2. Consider our motivating scenario about image analysis. Suppose that images are about nature (landscapes, seaside, etc.) Suppose that we create the following fuzzy knowledge base (Σ) in order to use reasoning-based image analysis:

$$\begin{aligned} \mathcal{T} &= \{\text{Leafs} \equiv \text{LightGreenColored} \sqcap \text{RegularTextured}, \\ &\quad \text{Log} \equiv \text{BrownColored} \sqcap \text{SmoothTextured}, \\ &\quad \text{Tree} \equiv \exists \text{hasPart} . (\text{Log} \sqcap \exists \text{isBelowOf} . \text{Leafs})\}, \\ \mathcal{R} &= \{\text{Trans}(\text{hasPart})\}. \end{aligned}$$

Subsequently, an image analysis/segmentation algorithm (like the RSST algorithm) has been applied and has segmented the image in several regions. The analysis algorithm also produces a set of values for each region, which consist of values about their color (in some color model), texture and shape (from various image specific transforms). These values can then be fuzzified with the aid of fuzzy partitions [20] creating fuzzy assertions like the following ones:

$$\begin{aligned} \mathcal{A} &= (o_1 : \text{LightGreenColored}) \geq 0.85, (o_1 : \text{RegularTextured}) \geq 0.7, \\ &\quad (o_2 : \text{BrownColored}) \geq 1.0, (o_2 : \text{SmoothTextured}) \geq 0.9, \\ &\quad ((o_1, o_2) : \text{isAboveOf}) \geq 0.9, ((o_3, o_2) : \text{hasPart}) \geq 0.8 \end{aligned}$$

In order for a fuzzy interpretation \mathcal{I} to be a model of \mathcal{T} it should hold that:

$$\begin{aligned} \text{Leafs}^{\mathcal{I}}(o_1^{\mathcal{I}}) &= t(\text{LightGreenColored}^{\mathcal{I}}(o_1^{\mathcal{I}}), \text{RegularTextured}^{\mathcal{I}}(o_1^{\mathcal{I}})) = t(0.85, 0.7). \\ \text{Log}^{\mathcal{I}}(o_2^{\mathcal{I}}) &= t(\text{BrownColored}^{\mathcal{I}}(o_2^{\mathcal{I}}), \text{SmoothTextured}^{\mathcal{I}}(o_2^{\mathcal{I}})) = t(1.0, 0.9). \end{aligned}$$

Moreover,

$$\begin{aligned} \text{Tree}^{\mathcal{I}}(o_3^{\mathcal{I}}) &= \sup_b \{t(\text{hasPart}^{\mathcal{I}}(o_3^{\mathcal{I}}, b), (\text{Log} \sqcap \exists \text{isBelowOf} . \text{Leafs})^{\mathcal{I}}(b))\} \\ &= \sup_b \{t(\text{hasPart}^{\mathcal{I}}(o_3^{\mathcal{I}}, b), t(\text{Log}^{\mathcal{I}}(b), \sup_c \{t(\text{isBelowOf}^{\mathcal{I}}(b, c), \text{Leafs}^{\mathcal{I}}(c))\}))\} \\ &\geq t(\text{hasPart}^{\mathcal{I}}(o_3^{\mathcal{I}}, o_2^{\mathcal{I}}), t(\text{Log}^{\mathcal{I}}(o_2^{\mathcal{I}}), t((\text{isAboveOf}^-)^{\mathcal{I}}(o_2^{\mathcal{I}}, o_1^{\mathcal{I}}), \text{Leafs}^{\mathcal{I}}(o_1^{\mathcal{I}})))) \\ &\geq t(0.8, t(t(1.0, 0.9), t(0.9, t(0.85, 0.7)))) \end{aligned}$$

Finally, depending on which t -norm we use in our application we can infer different values for $o_3^{\mathcal{I}}$ being a tree. For example, if t is the product t -norm then, $\text{Tree}^{\mathcal{I}}(o_3^{\mathcal{I}}) \geq 0.385$, if t is the Lukasiewicz t -norm then $\text{Tree}^{\mathcal{I}}(o_3^{\mathcal{I}}) \geq 0.15$, while if we use the Gödel t -norm then $\text{Tree}^{\mathcal{I}}(o_3^{\mathcal{I}}) \geq 0.7$.

3.2. Concept equivalences of fuzzy DLs

In crisp DLs the semantics of the language forces a number of concept equivalences to hold. For example, since in Boolean algebra the De Morgan laws are satisfied it holds that $\neg(C \sqcap D) \equiv \neg C \sqcup \neg D$. In the current section we will investigate the most common concept equivalences of crisp DLs in the context of fuzzy DLs. Several of these properties might have already been presented sparsely in various papers, either implicitly or explicitly. Other might be easily obtained by considering well known results in fuzzy First-Order Logic [25]. Here we attempt to gather the most common ones and explicitly present them in the DL setting, which we believe is beneficial for the wider probably interested but likely unfamiliar with fuzzy logic Semantic Web community.

Obviously, in the case of fuzzy DLs concept equivalences greatly depend on the mathematical properties of the fuzzy operators (norms) that are used each time. Hence, different combinations of norm operations result in f-DLs which satisfy different concept equivalences. For any triple $\langle c, t, u \rangle$, due to the standard properties of the fuzzy complement, t-norm and t-conorm, presented in Section 2.1, the following concept equivalences hold:

$$\begin{array}{ll} \neg \top & \equiv \perp, & \neg \perp & \equiv \top, \\ C \sqcap \top & \equiv C, & C \sqcup \perp & \equiv C, \\ C \sqcup \top & \equiv \top, & C \sqcap \perp & \equiv \perp. \end{array}$$

If the fuzzy complement is involutive then we also have, $\neg\neg C = C$. Now if the fuzzy triple satisfies the De Morgan laws (called *dual* triple), we additionally have,

$$\neg(C \sqcup D) \equiv \neg C \sqcap \neg D \text{ and } \neg(C \sqcap D) \equiv \neg C \sqcup \neg D.$$

For example the fuzzy triples, $\langle c_L, t_L, u_L \rangle$, $\langle c_L, t_G, u_G \rangle$, $\langle c_L, t_P, u_P \rangle$, are all dual triples. Moreover, for any dual triple $\langle c, t, u \rangle$ and S -implication \mathcal{J}_S the following hold:

$$\begin{array}{ll} \neg \exists R.C & \equiv \forall R.(\neg C), & \neg \forall R.C & \equiv \exists R.(\neg C), \\ \neg \leq pR & \equiv \geq (p+1)R, & \neg \geq pR & \equiv \begin{cases} \leq (p-1)R, & p \in \mathbb{N}^* \\ \perp, & p = 0 \end{cases} \end{array}$$

For example the quadruple $\langle c_L, t_G, u_G, \mathcal{J}_{KD} \rangle$, satisfies this equivalence. It is worth noting that the equivalences, $\exists R.C \equiv \neg \forall R. \neg C$ and $\forall R.C \equiv \neg \exists R. \neg C$ hold if additionally the fuzzy complement is involutive.

Additionally, if the fuzzy triple satisfies the laws of contradiction and excluded middle, then the following properties of boolean logic hold:

$$C \sqcap \neg C \equiv \perp \text{ and } C \sqcup \neg C \equiv \top.$$

These laws are quite hard to be satisfied by fuzzy triples. For example, from the above mentioned triples only the triple $\langle c_L, t_L, u_L \rangle$ satisfies these laws. Moreover, if triples satisfy the distributivity laws, then we have:

$$\begin{aligned} C_1 \sqcap (C_2 \sqcup C_3) &\equiv (C_1 \sqcap C_2) \sqcup (C_1 \sqcap C_3) \text{ and} \\ C_1 \sqcup (C_2 \sqcap C_3) &\equiv (C_1 \sqcup C_2) \sqcap (C_1 \sqcup C_3). \end{aligned}$$

For example the pair of operators t_G, u_G is *the only* one that satisfies these laws. Furthermore, under the semantics of number restrictions [11, 23] and used here, the concept equivalence $\exists R.\top \equiv \geq 1R$, holds. Concluding, we remark that it is known that no combination of fuzzy operators satisfies all the Boolean properties at the same time.

The above analysis justifies the need for introducing a special notation for distinguishing between fuzzy DLs that use different norm functions. For example, usually in fuzzy logic [25] the name of the fuzzy implication is used to denote the implication operator considered in the specific setting, while the other operators are assumed to be the defined ones. For example, in case \mathcal{J} is an S -implication the fuzzy complement and t-conorm are also defined, since $\mathcal{J}(a, b) = u(c(a), b)$, while t is obtained by $t(a, b) = c(u(c(a), c(b)))$, while if \mathcal{J} is an R -implication then the t-norm is known, u is obtained as before but dually, while the precomplement of \mathcal{J} is taken as the fuzzy complement. Here we propose to use the name of the fuzzy implication as an index, more precisely the notation $f_{\mathcal{J}}$ -DL in order to distinguish that this is not another DL operator (as capital letters in DL notation indicate DL constructors). Hence, the notation f_{KD} - $SHOIN(\mathbf{D})$ indicates the fuzzy $SHOIN(\mathbf{D})$ language which uses the Lukasiewicz complement, the Gödel t-norm and t-conorm and the Kleene-Dienes fuzzy implication, while by f_L - $SHOIN(\mathbf{D})$ we indicate the fuzzy $SHOIN(\mathbf{D})$ language which uses the Lukasiewicz, t-norm, t-conorm, negation and fuzzy implication. Moreover, we use f_S - $SHOIN(\mathbf{D})$ to denote the whole family of f - $SHOIN(\mathbf{D})$ DLs that use S -implications and f_R - $SHOIN(\mathbf{D})$ for the family of R -implications. Furthermore, this notation is more modular in the following sense. Sometimes, the precomplement of a fuzzy implication is the Gödel complement, for which no reasoning algorithm exists. Thus, one usually replaces it with a specific continuous and involutive complement. In this case we can use the notation, $f_{\langle c_L, t_p, u_p, \mathcal{J}_p \rangle}$ - \mathcal{L} to explicitly define the fuzzy operators used.

4. A Simplistic Fuzzy Extension of OWL

In this section, based on our simplistic fuzzy- $SHOIN(\mathbf{D})$ extension we similarly present a simplistic fuzzy extension of OWL DL by adding degrees to OWL facts. We will present the model-theoretic semantics of the extended language and the syntactic changes that need to take place both in the abstract syntax as well as the RDF/XML syntax of the extended language.

4.1. Syntax and semantics of f -OWL

Our fuzzy OWL language shares essentially the same syntax with the crisp OWL language as presented in Section 2.3. Hence, one is able to create OWL class descriptions and OWL class and property axioms in exactly the same way this is done in the OWL

Table 5: Fuzzy OWL Class and Property Descriptions

Abstract Syntax	DL Syntax	Semantics
Class(A) owl:Thing owl:Nothing	A \top \perp	$A^{\mathcal{I}} : \Delta^{\mathcal{I}} \rightarrow [0, 1]$ $\top^{\mathcal{I}}(a) = 1$ $\perp^{\mathcal{I}}(a) = 0$
intersectionOf($C D \dots$) unionOf($C D \dots$) complementOf(C) oneOf($o_1 o_2 \dots$)	$C \sqcap D$ $C \sqcup D$ $\neg C$ $\{o_1\} \sqcup \dots \sqcup \{o_k\}$	$(C \sqcap D)^{\mathcal{I}}(a) = t(C^{\mathcal{I}}(a), D^{\mathcal{I}}(a))$ $(C \sqcup D)^{\mathcal{I}}(a) = u(C^{\mathcal{I}}(a), D^{\mathcal{I}}(a))$ $(\neg C)^{\mathcal{I}}(a) = c(C^{\mathcal{I}}(a))$ $(\{o_1\} \sqcup \dots \sqcup \{o_k\})^{\mathcal{I}}(a) = \begin{cases} 1, & a \in \{o_1^{\mathcal{I}}, \dots, o_k^{\mathcal{I}}\} \\ 0, & \text{otherwise} \end{cases}$
restriction(R someValuesFrom(C)) restriction(R allValuesFrom(C)) restriction(R value(o)) restriction(R minCardinality(p)) restriction(R maxCardinality(p)) restriction(R cardinality(p))	$\exists R.C$ $\forall R.C$ $\exists R.\{o\}$ $\geq pR$ $\leq pR$ $\geq pR \sqcap \leq pR$	$(\exists R.C)^{\mathcal{I}}(a) = \sup_{b \in \Delta^{\mathcal{I}}} t(R^{\mathcal{I}}(a, b), C^{\mathcal{I}}(b))$ $(\forall R.C)^{\mathcal{I}}(a) = \inf_{b \in \Delta^{\mathcal{I}}} \mathcal{J}(R^{\mathcal{I}}(a, b), C^{\mathcal{I}}(b))$ $(\exists R.\{o\})^{\mathcal{I}}(a) = R^{\mathcal{I}}(a, o^{\mathcal{I}})$ $(\geq pR)^{\mathcal{I}}(a) = \sup_{b_1, \dots, b_p \in \Delta^{\mathcal{I}}} t(\prod_{i=1}^p R^{\mathcal{I}}(a, b_i), \prod_{i < j} \{b_i \neq b_j\})$ $(\leq pR)^{\mathcal{I}}(a) = \inf_{b_1, \dots, b_{p+1} \in \Delta^{\mathcal{I}}} \mathcal{J}(\prod_{i=1}^{p+1} R^{\mathcal{I}}(a, b_i), \prod_{i < j} \{b_i = b_j\})$ $(\geq pR \sqcap \leq pR)^{\mathcal{I}}(a) = t((\geq pR)^{\mathcal{I}}(a), (\leq pR)^{\mathcal{I}}(a))$
oneOf($c_1 c_2 \dots$) restriction(T someValuesFrom(d)) restriction(T allValuesFrom(d)) restriction(T value(c)) restriction(T minCardinality(p)) restriction(T maxCardinality(p)) restriction(T cardinality(p))	$\{c_1\} \sqcup \dots \sqcup \{c_k\}$ $\exists T.d$ $\forall T.d$ $\exists T.\{c\}$ $\geq pT$ $\leq pT$ $\geq pT \sqcap \leq pT$	$(\{c_1\} \sqcup \dots \sqcup \{c_k\})^{\mathcal{I}}(y) = \begin{cases} 1, & y \in \{c_1^{\mathcal{D}}, \dots, c_k^{\mathcal{D}}\} \\ 0, & \text{otherwise} \end{cases}$ $(\exists T.d)^{\mathcal{I}}(a) = \sup_{y \in \Delta^{\mathcal{D}}} t(T^{\mathcal{I}}(a, y), d^{\mathcal{I}}(y))$ $(\forall T.d)^{\mathcal{I}}(a) = \inf_{y \in \Delta^{\mathcal{D}}} \mathcal{J}(T^{\mathcal{I}}(a, y), d^{\mathcal{I}}(y))$ $(\exists T.\{c\})^{\mathcal{I}}(a) = T^{\mathcal{I}}(a, c^{\mathcal{D}})$ $(\geq pT)^{\mathcal{I}}(a) = \sup_{y_1, \dots, y_p \in \Delta^{\mathcal{D}}} t(\prod_{i=1}^p T^{\mathcal{I}}(a, y_i), \prod_{i < j} \{y_i \neq y_j\})$ $(\leq pT)^{\mathcal{I}}(a) = \inf_{y_1, \dots, y_{p+1} \in \Delta^{\mathcal{D}}} \mathcal{J}(\prod_{i=1}^{p+1} T^{\mathcal{I}}(a, y_i), \prod_{i < j} \{y_i = y_j\})$ $(\geq pT \sqcap \leq pT)^{\mathcal{I}}(a) = t((\geq pT)^{\mathcal{I}}(a), (\leq pT)^{\mathcal{I}}(a))$

language. For example one can provide a visual description of the concept Mountain as something that is brown, big and coarse. In OWL abstract syntax [2] this definition can look as, $\text{Class}(\text{Mountain complete intersectionOf}(\text{Brown Coarse Big}))$.

The differences between crisp OWL and fuzzy OWL raise in the definition of OWL facts (individual axioms) in order to be able to specify the membership degree and the type of inequality of an individual (pair of individuals) to a fuzzy class (property). We refer to such axioms as *fuzzy facts*. For example, in the previous case one might want to state that an image region, reg_1 , is brown to a degree greater or equal than 0.8. As we will see in the following in f-OWL the abstract syntax of such an axiom is, $\text{Individual}(reg_1 \text{ type}(\text{Brown}) \geq 0.8)$.

Although, the syntax modifications are minor, the semantics of f-OWL are based on fuzzy interpretations, in order to interpret OWL classes and properties as fuzzy sets and fuzzy relations. In the case of f-OWL DL these interpretations are fairly standard by description logic standards. Hence, as introduced in Section 3, a fuzzy interpretation is a

4-tuple $\mathcal{I} = (\Delta^{\mathcal{I}}, \Delta_{\mathbf{D}}, \cdot^{\mathcal{I}}, \cdot^{\mathbf{D}})$ where $\Delta^{\mathcal{I}}$, $\Delta_{\mathbf{D}}$, $\cdot^{\mathcal{I}}$ and $\cdot^{\mathbf{D}}$ are as in the case of $\text{f-SHOIN}(\mathbf{D})$.

An f-OWL interpretation can be extended to give semantics to fuzzy class descriptions and fuzzy class and fuzzy property axioms. The abstract syntax, the respective fuzzy DL syntax and the semantics of f-OWL class descriptions are depicted in Table 5. The abstract syntax, f-DL syntax and semantics of f-OWL class and property axioms are depicted in Table 6. In Table 6 the notation $([\dots])$ is used to indicate that a field is optional. Hence, specifying a membership degree along with an inequality is optional. This will be further explained in the next section.

A *fuzzy ontology*, O , is a set of f-OWL axioms. We say that a fuzzy interpretation \mathcal{I} is a model of O iff it satisfies all axioms in O . A fuzzy ontology O_1 *entails* a fuzzy ontology O_2 , written $O_1 \models O_2$ if every model of O_1 is a model of O_2 .

4.2. Syntactic Sugar Constructors of Fuzzy OWL

One of the interesting modelling properties of OWL is that it tries to abstract from DL notation and axioms providing ways for even inexperienced users to create ontologies. For those reasons OWL offers a set of axioms that are actually syntactic sugar of Description Logic axioms. For example, OWL offers the ability to declare the range of a property R by hiding the cumbersome DL syntax. Many such examples we have already seen before. In classical logics the translation from OWL axioms into DL axioms is generally considered to be straightforward and actually there exist more than one ways to map an OWL axiom into a DL axiom, since due to the properties of Boolean algebra several equivalences exist. On the other hand, the case is quite different for fuzzy OWL and fuzzy DLs. This is obvious since as we have already discussed not all concept equivalences hold in fuzzy DLs, thus different ways to translate an f-OWL axiom can lead to different semantic meanings. In the following we will provide an investigation of the semantics of fuzzy OWL's syntactic sugar axioms and we will discuss in which cases these different ways of modelling coincide.

4.2.1. Domain and Range Restrictions

As we have already seen in Section 2.3 property domain axiom are usually translated into DL axioms of the form $\exists R.\top \sqsubseteq C$, which mean that if $\langle a, b \rangle \in R^{\mathcal{I}}$ then $a \in C^{\mathcal{I}}$. If we use fuzzy semantics then if \mathcal{I} is a fuzzy interpretation then \mathcal{I} satisfies $\exists R.\top \sqsubseteq C$ if $\sup_{c \in \Delta^{\mathcal{I}}} t(R^{\mathcal{I}}(a, c), 1) \leq C^{\mathcal{I}}(a)$. Hence, for an arbitrary $b \in \Delta^{\mathcal{I}}$ and due to the boundary condition of t -norms $t(R^{\mathcal{I}}(a, b), 1) = R^{\mathcal{I}}(a, b) \leq C^{\mathcal{I}}(a)$. We see that provides a quite intuitive interpretation, i.e. that the degree that a belongs to $C^{\mathcal{I}}$ is at least equal to the degree that the relation $R^{\mathcal{I}}(a, b)$ holds.

The case of range restrictions is more involved. In classical DLs two different but equivalent translations can be used. On the one hand $\text{ObjectProperty}(R \text{ range}(C))$ can be transformed into $\top \sqsubseteq \forall R.C$, while on the other into $\exists R^{\neg}.\top \sqsubseteq C$. In the case of fuzzy DLs these two different axioms do not always give the same intuitive meaning. Let \mathcal{I} be a fuzzy interpretation. As with domain restrictions the second axiom will finally give the inequality $(R^{\neg})^{\mathcal{I}}(a, b) = R^{\mathcal{I}}(b, a) \leq C^{\mathcal{I}}(a)$ which is again quite intuitive. On the other hand

Table 6: Fuzzy OWL Axioms

Abstract Syntax	DL Syntax	Semantics
Class(A partial $C_1 \dots C_n$)	$A \sqsubseteq \prod_{i=1}^n C_i$	$A^{\mathcal{I}}(a) \leq t(C_1^{\mathcal{I}}(a), \dots, C_n^{\mathcal{I}}(a))$
Class(A complete $C_1 \dots C_n$)	$A \equiv \prod_{i=1}^n C_i$	$A^{\mathcal{I}}(a) = t(C_1^{\mathcal{I}}(a), \dots, C_n^{\mathcal{I}}(a))$
EnumeratedClass(A $o_1 \dots o_k$)	$A \equiv \{o_1, \dots, o_k\}$	$A^{\mathcal{I}}(a) = 1$ if $a \in \{o_1^{\mathcal{I}}, \dots, o_k^{\mathcal{I}}\}$, $A^{\mathcal{I}}(a) = 0$ otherwise
SubClassOf(C_1, C_2)	$C_1 \sqsubseteq C_2$	$C_1^{\mathcal{I}}(a) \leq C_2^{\mathcal{I}}(a)$
EquivalentClasses($C_1 \dots C_n$)	$C_1 \equiv \dots \equiv C_n$	$C_1^{\mathcal{I}}(a) = \dots = C_n^{\mathcal{I}}(a)$
DisjointClasses($C_1 \dots C_n$)	$C_i \sqcap C_j \sqsubseteq \perp$ $C_i \sqsubseteq \neg C_j$	$t(C_i^{\mathcal{I}}(a), C_j^{\mathcal{I}}(a)) = 0$, $1 \leq i < j \leq n$ $C_i^{\mathcal{I}}(a) \leq (\neg C_j)^{\mathcal{I}}(a)$, $1 \leq i < j \leq n$
SubPropertyOf(R_1, R_2)	$R_1 \sqsubseteq R_2$	$R_1^{\mathcal{I}}(a, b) \leq R_2^{\mathcal{I}}(a, b)$
EquivalentProperties($R_1 \dots R_n$)	$R_1 \equiv \dots \equiv R_n$	$R_1^{\mathcal{I}}(a, b) = \dots = R_n^{\mathcal{I}}(a, b)$
ObjectProperty(R super(R_1) ... super(R_n))	$R \sqsubseteq R_i$	$R^{\mathcal{I}}(a, b) \leq R_i^{\mathcal{I}}(a, b)$
domain(C_1) ... domain(C_k)	$\exists R. \top \sqsubseteq C_i$	$R^{\mathcal{I}}(a, b) \leq C_i^{\mathcal{I}}(a)$
range(C_1) ... range(C_h)	$\top \sqsubseteq \forall R. C_i$	$1 \leq \mathcal{J}(R^{\mathcal{I}}(a, b), C_i^{\mathcal{I}}(b))$
[InverseOf(S)]	$R \equiv S^{-}$	$R^{\mathcal{I}}(a, b) = (S^{-})^{\mathcal{I}}(a, b)$
[Symmetric]	$R \equiv R^{-}$	$R^{\mathcal{I}}(a, b) = (R^{-})^{\mathcal{I}}(a, b)$
[Functional]	$\top \sqsubseteq \leq 1R$	$\inf_{b_1, b_2 \in \Delta^{\mathcal{I}}} \mathcal{J}(\bigwedge_{i=1}^2 R^{\mathcal{I}}(a, b_i), b_1 = b_2) \geq 1$
	Func(R)	$R^{\mathcal{I}}(a, b_i) > 0, R^{\mathcal{I}}(a, b_j) > 0 \rightarrow b_i = b_j$
[InverseFunctional]	$\top \sqsubseteq \leq 1R^{-}$	$\inf_{b_1, b_2 \in \Delta^{\mathcal{I}}} \mathcal{J}(\bigwedge_{i=1}^2 (R^{-})^{\mathcal{I}}(a, b_i), b_1 = b_2) \geq 1$
	Func(R^{-})	$R^{\mathcal{I}}(b_i, a) > 0, R^{\mathcal{I}}(b_j, a) > 0 \rightarrow b_i = b_j$
[Transitive]	Trans(R)	$\sup_{b \in \Delta^{\mathcal{I}}} t(R^{\mathcal{I}}(a, b), R^{\mathcal{I}}(b, c)) \leq R^{\mathcal{I}}(a, c)$
SubPropertyOf(T_1, T_2)	$T_1 \sqsubseteq T_2$	$T_1^{\mathcal{I}}(a, y) \leq T_2^{\mathcal{I}}(a, y)$
EquivalentProperties($T_1 \dots T_n$)	$T_1 \equiv \dots \equiv T_n$	$T_1^{\mathcal{I}}(a, y) = \dots = T_n^{\mathcal{I}}(a, y)$
ObjectProperty(T super(T_1) ... super(T_n))	$T \sqsubseteq T_i$	$T^{\mathcal{I}}(a, y) \leq T_i^{\mathcal{I}}(a, y)$
domain(C_1) ... domain(C_k)	$\exists T. \top \sqsubseteq C_i$	$T^{\mathcal{I}}(a, y) \leq C_i^{\mathcal{I}}(a)$
range(d_1) ... range(d_h)	$\top \sqsubseteq \forall T. d_i$	$1 \leq \inf_{y \in \Delta_{\mathbf{D}}} \mathcal{J}(R^{\mathcal{I}}(a, y), d_i^{\mathcal{I}}(y))$
[Functional]	$\top \sqsubseteq \leq 1T$	$\inf_{y_1, y_2 \in \Delta_{\mathbf{D}}} \mathcal{J}(\bigwedge_{i=1}^2 R^{\mathcal{I}}(a, y_i), y_1 = y_2) \geq 1$
	Func(T)	$T^{\mathcal{I}}(a, y_i) > 0, T^{\mathcal{I}}(a, y_j) > 0 \rightarrow y_i = y_j$
Individual(o type(C_1) $[\bowtie]$ $[n_1]$... type(C_m) $[\bowtie]$ $[n_\ell]$)	$(o : C_i) \bowtie n_i$	$C_i^{\mathcal{I}}(o^{\mathcal{I}}) \bowtie n_i, m_i \in [0, 1], 1 \leq i \leq \ell$
value(R_1, o_1) $[\bowtie]$ $[n_1]$... value(R_ℓ, o_ℓ) $[\bowtie]$ $[n_\ell]$)	$((o, o_i) : R_i) \bowtie n_i$	$R_i^{\mathcal{I}}(o^{\mathcal{I}}, o_i^{\mathcal{I}}) \bowtie n_i, n_i \in [0, 1], 1 \leq i \leq \ell$
value(T_1, c_1) $[\bowtie]$ $[n_1]$... value(T_ℓ, c_ℓ) $[\bowtie]$ $[n_\ell]$)	$((o, c_i) : T_i) \bowtie n_i$	$T_i^{\mathcal{I}}(o^{\mathcal{I}}, c_i^{\mathcal{I}}) \bowtie n_i, n_i \in [0, 1], 1 \leq i \leq \ell$
Sameindividual($o_1 \dots o_\ell$)	$o_i \doteq o_j$	$o_i^{\mathcal{I}} = o_j^{\mathcal{I}}, 1 \leq i < j \leq \ell$
DifferentIndividuals($o_1 \dots o_\ell$)	$o_i \neq o_j$	$o_i^{\mathcal{I}} \neq o_j^{\mathcal{I}}, 1 \leq i < j \leq \ell$

the first axiom gives $1 \leq \inf_{c \in \Delta^{\mathcal{I}}} \mathcal{J}(R^{\mathcal{I}}(a, c), C^{\mathcal{I}}(c))$. If \mathcal{J} is an R -implication, then by the properties of R -implications introduced in Section 2.1 and for some arbitrary $b \in \Delta^{\mathcal{I}}$ we obtain $R^{\mathcal{I}}(a, b) \leq C^{\mathcal{I}}(b)$, which coincides with the semantics of the first axiom. But, in case \mathcal{J} is an S -implication no such equivalence can be derived. Needless to say that depending on the norm operators used, such an axiom might cause R and C to be interpreted as crisp sets. For example for the Kleene-Dienes S -implication, $\max(1 - R^{\mathcal{I}}(a, d), C^{\mathcal{I}}(d)) \geq 1$ iff either $R^{\mathcal{I}}(a, d) = 0$ or $C^{\mathcal{I}}(d) = 1$. Consequently, it seems that the second translation gives more intuitive semantics regardless of the fuzzy implication used.

4.2.2. Functional Role Axioms

According to Table 2.3 an OWL functional role axiom of the form `ObjectProperty(R Functional)` is translated into the DL axiom $\top \sqsubseteq \leq 1R$. Intuitively, this means that all objects of $\Delta^{\mathcal{I}}$ participate in $R^{\mathcal{I}}$ with at-most one other object. In the fuzzy case this axiom gives us the inequation $\inf_{b_1, b_2 \in \Delta^{\mathcal{I}}} \mathcal{J}(\overset{2}{t} R^{\mathcal{I}}(a, b_i), b_1 = b_2) \geq 1$. Since we only consider crisp equalities and inequalities of objects (i.e. no similarity measures), in order for this inequation to hold we should either have that $t(R^{\mathcal{I}}(a, b_1), R^{\mathcal{I}}(a, b_2)) = 0$ or $b_1 = b_2$ for arbitrary $b_1, b_2 \in \Delta^{\mathcal{I}}$. For *non-nilpotent* t -norms the first implies that either $R^{\mathcal{I}}(a, b_1) = 0$ or $R^{\mathcal{I}}(a, b_2) = 0$. In other words R is functional if for every $a \in \Delta^{\mathcal{I}}$ there exists at-most one $c \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(a, c) > 0$. On the other hand for nilpotent t -norms it is possible that there are many c_i such that $R^{\mathcal{I}}(a, c_i) > 0$ and $t(R^{\mathcal{I}}(a, c_i), R^{\mathcal{I}}(a, c_j)) = 0, i \neq j$ as long as the degrees $R^{\mathcal{I}}(a, c_i)$ are small enough for the t -norm to be equal to 0. This effect was first observed for fuzzy DLs under the Lukasiewicz operators (to which all other nilpotent t -norms are isomorphic) in [33]. Nevertheless, we note that this should not be considered as a problematic or counterintuitive case but rather as a feature of the fuzzy semantics provided by such operators. Certainly, a user familiar with classical logics would probably not want to have this effect, but in a fuzzy setting, this behavior (as the one that we will see about disjointness axioms next) might be acceptable. Concluding, in order to provide the ability to use either of these semantics we have also given two different translations for functional role axioms the one of which directly uses the DL axiom `Func(R)`.

4.2.3. Disjointness Axioms

Now we investigate on disjointness axioms. Observe that we have also given two semantics for disjoint classes. These are based on the two different syntactic forms for representing concept disjointness in classical Description Logics, namely $C \sqcap D \sqsubseteq \perp$ and $C \sqsubseteq \neg D$. But, while in crisp DLs the semantics of these two syntactic forms coincide, this is not *always* true in fuzzy DLs [15]. More precisely, we have the following result.

Lemma 4.1. *Let $f_{\langle c, t, u, \mathcal{J} \rangle}$ -OWL, such that $\langle c, t, u \rangle$ satisfy the law of contradiction. Then $C \sqcap D \sqsubseteq \perp$, holds if and only if $C \sqsubseteq \neg D$, holds.*

Proof: The proof is given in the appendix ■

In case where the law of contradiction does not hold, these definitions have completely different meanings. Consider for example an axiom of the form `DisjointClasses(C D)`. Using the first form of Table 6 we have $t(C^{\mathcal{I}}(a), D^{\mathcal{I}}(a)) = 0$ for all $a \in \Delta^{\mathcal{I}}$. Now if t is a non-nilpotent t -norm we have that $t(C^{\mathcal{I}}(a), D^{\mathcal{I}}(a)) = 0$ iff either $C^{\mathcal{I}}(a) = 0$ or $D^{\mathcal{I}}(a) = 0$. This means that C and D do not “share” any objects even not to a very small degree. On the other hand with the second definition we have that $C^{\mathcal{I}}(a) \leq c(D^{\mathcal{I}}(a))$ and if for example c is the Lukasiewicz negation we get $C^{\mathcal{I}}(a) \leq 1 - D^{\mathcal{I}}(a) \Rightarrow C^{\mathcal{I}}(a) + D^{\mathcal{I}}(a) \leq 1$. In other words C and D are considered disjoint even if they share some objects but as long as these objects do not “strongly” belong (i.e. to a high degree) to both of them. Concluding, as we can see the first definition gives a more crisp notion of disjointness which is closer to our usual intuition, but the second one could be seen as a fuzzy notion of disjointness.

4.2.4. The one-of/enumeration constructor

Finally, as in the case for crisp DLs, we can easily note that both the one-of/enumeration constructor as well as the `hasValue/fills` operators are syntactic sugar in the presence of *standard* nominals, disjunction and existential quantification. More precisely, $(\{o_1\} \sqcup \dots \sqcup \{o_k\})^{\mathcal{I}}(a) = u(\{o_1\}^{\mathcal{I}}(a), \dots, \{o_k\}^{\mathcal{I}}(a))$ and due to the boundary condition of t -conorms and the interpretation of nominals, $u(\{o_1\}^{\mathcal{I}}(a), \dots, \{o_k\}^{\mathcal{I}}(a)) = 1$ iff there exists at least one $j \in [1, k]$ such that $o_j^{\mathcal{I}} = a$ or stated otherwise, if $a \in \{o_1, \dots, o_k\}$ which gives the semantics of one-of. On the other for `hasValue` restrictions, $\exists R.\{o\}$ is interpreted as $(\exists R.\{o\})^{\mathcal{I}}(a) = \sup_{b \in \Delta^{\mathcal{I}}} t(R^{\mathcal{I}}(a, b), \{o\}^{\mathcal{I}}(b))$, but due to the semantics of nominals and the boundary conditions of t -norms the right-hand side can be simplified into $R^{\mathcal{I}}(a, o^{\mathcal{I}})$ since in a different case (i.e. $b \neq o^{\mathcal{I}}$) $t(R^{\mathcal{I}}(a, b), 0) = 0$ which coincides with the semantics of the `fills` constructor we have presented.

4.3. Abstract and concrete RDF syntax of f -OWL

In the previous section we showed that in order to represent membership degrees in fuzzy OWL the abstract syntax has to be extended. More precisely, from Table 6 we can see that the abstract syntax of the one-of constructor and the OWL individual axioms has been extended. In the current section we make these extensions formal as well as show how one could use this abstract syntax to provide an RDF/XML concrete syntax by which we can concretely represent fuzzy knowledge in real ontologies.

Table 7 presents the abstract syntax of fuzzy facts. We see that the usual definition of these OWL constructors is extended with the new elements *degree* and *ineqType*. The element `ineqType` is used to specify the inequality type that is taking place in the instance relation. Thus, the values of this element are the strings, “>=”, “<=”, “>” and “<” and “=”. Finally, the element `degree` is used to specify the membership degree, taken from the interval $[0, 1]$ that the specified instance relation holds.

As we can see the new elements are optional, i.e. the user might not specify either an inequality type or a membership degree for the instance relation. In that case it is reasonable to consider by default that the inequality type is of the form = and the membership

Table 7: Abstract Syntax of f-OWL

individual	::=	'Individual(' [individualID] {annotation} {'type'(type ')} [ineqType] [degree]} {value [ineqType] [degree] } ')
ineqType	::=	'=' '>=' '>' '<=' '<'
degree	::=	real-number-between-0-and-1-inclusive

degree is equal to 1. Moreover, we see that these elements are placed both after the type element as well as after the value element in the definition of individual axioms. In the former case we can specify the membership of fuzzy facts involving an individual and a concept, while in the latter we can specify the membership between a pair of individuals and a fuzzy role.

Besides the abstract syntax that is intended to be a human readable form of OWL axioms close to that of DLs, OWL also offers an XML like syntax for representing actual knowledge and axioms in a concrete form. This syntax follows the ideas of the RDF/XML syntax that has been proposed for RDF⁷ [34]. Using our previously extended abstract syntax we can similarly extend the RDF/XML syntax of OWL in order to represent fuzzy information. In the following we will mainly use some examples to illustrate how such a syntax could look.

As with classical OWL and RDF there are two different ways by which facts (individual axioms) can be encoded. First we can use the abbreviating syntax of RDF/XML for specifying instance relations. Then, for example the RDF/XML syntax for representing the fact that Rome is hot to a degree at least 0.7 and close to Athens to a degree exactly 0.65 could look like the following:

```
<Hot rdf:about="Rome" owlx:ineqType="≥" owlx:degree="0.7">
  <isCloseTo rdf:resource="Athens" owlx:degree="0.65"/>
</HotPlace>
```

where we are using the new elements `owlx:ineqType` and `owlx:degree`. On the other hand we could also use the RDF element `rdf:Description` to provide a different RDF/XML form. In this case RDF/XML syntax looks as follows:

```
<rdf:Description rdf:about="reg-1">
  <rdf:type rdf:resource="Blue" owlx:ineqType="≥" owlx:degree="0.9"/>
  <isOverlappingWith rdf:resource="reg-2" owlx:ineqType="≥" owlx:degree="0.75"/>
</rdf:Description>
```

⁷RDF is another ontology language, weaker than OWL and intended for representing basic ontologies. Roughly speaking it allows for domain and range restrictions, role inclusion axioms and concept subsumptions.

Concluding, we want to stress out again that the intention of this section is not to provide the commonly agreed or best way to represent vague information using OWL. This could only be the result of extensive debates between people from different communities and taking into account different requirements and ideas. Our work should be understood as a proof of concept for providing means to represent vague information by *extending* the RDF/XML syntax. Furthermore, it could serve as a guideline if a fuzzy OWL standardization group is ever realized. There are actually several other ways to represent vague information in Semantic Web languages. For example, information could be stored in the form of annotations or as simply as comments, thus avoiding the burden of extending the language. This has been done in [35] for representing probabilistic information in OWL 2 (a forthcoming extension of OWL) and in [36] for representing fuzzy information in SPARQL (a query language for RDF). Moreover, there are proposals for using the standard building blocks of the language. For example, in [37] the authors use datatype properties to store fuzzy information in fuzzy RDF. More precisely, one could define a property like `hasDegree` of type `float` and a property `hasIneqType` of type `string` and use them in the obvious way in individual axioms. In the same spirit the authors in [38] propose the use of the properties `membershipOf`, `moreOrEquivalent`, etc. to represent fuzzy assertions.

In general it is not easy to assess which method is the best, because each one of them has its pros and cons. For example, with annotations and comments one provides a semanticless way of representing semantic information. Furthermore, new tools and parsers need to be implemented which will decompose the annotations and convert them in fuzzy assertions, but on the other hand there is compatibility with classical tools (parsers, reasoners) which can simply ignore them and reason as if all degrees were 0 or 1. Regarding using the building blocks of the language, one is usually unable to avoid unwanted semantic effects. For example, in [38] the authors treat both concepts and roles as individuals (an effect known as meta-modelling, i.e. OWL Full) in order to be able to represent a ternary relation such as a fuzzy assertion (i.e. `assertion(a, C, n)`). This, certainly destroys compatibility with crisp reasoners, but existing parsers can be used while only conversion tools implemented. Certainly in the absence of a standard it is relatively difficult for everyone to agree upon which syntax to use.

5. From f-OWL Entailment to Fuzzy DL Satisfiability

One of the two major goals of the current paper is to present a way by which f-OWL inference problems and ontologies can be reduced to f- $SHOIN(\mathbf{D})$ inference problems and knowledge bases. By this way one could support reasoning in f-OWL by using already known or implemented reasoning algorithms for fuzzy DLs. Hence, in the current section we will show how to reduce f-OWL entailment to f- $SHOIN(\mathbf{D})$ satisfiability. As described in [4] for crisp OWL and $SHOIN(\mathbf{D})$ this process involves two steps. In the first step OWL is translated into $SHOIN(\mathbf{D})$ thus translating entailment between OWL ontologies into entailment between $SHOIN(\mathbf{D})$ knowledge bases. Subsequently, since knowledge

base entailment is not a standard DL inference service this should be further reduced to knowledge base satisfiability.

5.1. From f-OWL to f-SHOIN(D)

The reduction of OWL class and property descriptions and OWL class and property axioms can be simply defined by an inductive function over the mappings between f-OWL abstract syntax and the respective f-DL syntax, as these have been shown in Tables 5 and 6. For example if \mathcal{V} is the function then an axiom of the form $\text{Class}(A \text{ partial } C_1 C_2 \dots C_n)$ is mapped through \mathcal{V} to $A \sqsubseteq \mathcal{V}(C_1) \sqcap \mathcal{V}(C_2) \sqcap \dots \sqcap \mathcal{V}(C_n)$, and subsequently \mathcal{V} again inductively reduces every OWL class description $C_i, 1 \leq i \leq n$ into an f-SHOIN(D)-concept using the mappings between class descriptions and DL classes of Table 5. Actually, this reduction is identical to the one provided in [4] since the syntactic extensions of the simplistic extension are only limited to (fuzzy) instance relations.

The aforementioned part of the translation is relatively straightforward. The most complex part identified in [4], is the translation of individual axioms (facts) because they can be stated with respect to anonymous individuals. In [4] two translations were provided, one for OWL DL and one for OWL Lite. This is because the translation of OWL DL uses nominals which OWL Lite does not support. By closely inspecting the abstract syntax of fuzzy individual axioms, from Table 6, and the translations in [4], would reveal that the OWL Lite reduction serves better our needs in the fuzzy case even when we consider the reduction of OWL DL facts. This is due to the presence of inequality types and membership degrees. Table 8 defines a mapping (\mathcal{F}) that transforms OWL facts to f-SHOIN(D) assertions.

Table 8: From f-OWL facts to f-DL fuzzy assertions

f-OWL fragment F	Translation $\mathcal{F}(F)$
Individual($x_1 \bowtie n_1 \dots x_p \bowtie n_p$) $a : \text{type}(C) \bowtie n$ $a : \text{type}(C)$ $a : \text{value}(R x) \bowtie n$ $a : \text{value}(R x)$ $a : o$	$\mathcal{F}(a : x_1 \bowtie n_1), \dots, \mathcal{F}(a : x_n \bowtie n_p)$ where a is new $(a : \mathcal{V}(C)) \bowtie n$ $(a : \mathcal{V}(C)) = 1$ $((a, b) : R) \bowtie n, \mathcal{F}(b : x)$ where b is new $((a, b) : R) = 1, \mathcal{F}(b : x)$ where b is new $a = o$
Sameindividual($o_1 \dots o_n$)	$\mathcal{V}(o_i) = \mathcal{V}(o_j) \ 1 \leq i < j \leq n$
DifferentIndividuals($o_1 \dots o_n$)	$\mathcal{V}(o_i) \neq \mathcal{V}(o_j) \ 1 \leq i < j \leq n$

Consider for example, the fact $\text{Individual}(\text{type}(C) \text{ value}(R \text{ Individual}(\text{type}(D) \geq 0.8)) > 0.7)$. If we apply the mapping \mathcal{F} this fuzzy fact is translated to the fuzzy assertions $(a : \mathcal{V}(C)) = 1, ((a, b) : R) > 0.7$ and $(b : \mathcal{V}(D)) \geq 0.8$, where a and b are new individuals. First, we see that the fuzzy fact is stated with respect to two anonymous individuals. Hence, we have created two new individuals, a and b , which are used in the fuzzy assertions. Moreover,

Table 9: From entailment to unsatisfiability

Axiom A	Transformation $\mathcal{G}(A)$
$(a : C) \bowtie n$	$(a : C) \neg \bowtie n$
$((a, b) : R) \bowtie n$	$\{(a : \forall R. \neg B) + \bowtie u(c(n), c(n)), (b : B) \bowtie n\}$, where B is a new concept, for the case of S -implications, i.e. $f_S\text{-SHOIN}(\mathbf{D})$ or $\{(a : \forall R. \neg B) \geq 1, (b : \neg B) \neg \bowtie n\}$, where B is a new concept, for the $f_R\text{-SHOIN}(\mathbf{D})$ family of DLs or $\{(a : \exists R. B) \neg \bowtie n, (b : B) \geq 1\}$ where B is a new concept
$a \doteq b$	$a \not\equiv b$
$a \not\equiv b$	$a \doteq b$
$C \sqsubseteq D$	$\{(x : C) \geq n, (x : D) < n\}, \forall n \in [0, 1]$
$\text{Trans}(R)$	$\{(x : \exists R. (\exists R. \{y\})) \geq n, (x : \exists R. \{y\}) < n\}, \forall n \in [0, 1]$
$R \sqsubseteq S$	$\{(x : \exists R. \{y\}) \geq n, (x : \exists S. \{y\}) < n\}, \forall n \in [0, 1]$
$\neg \bowtie$ is the <i>negation</i> of \bowtie , e.g. if $\bowtie = \geq$, then $\neg \bowtie = <$, while $+\bowtie$ is the <i>reflected negation</i> of \bowtie , e.g. if $\bowtie = \geq$, then $+\bowtie = >$	

since there is no inequality type and membership degree specified for the membership of the anonymous individual to concept C we have used the default ones, which is the equality and the degree 1.

Theorem 5.1. *The translation from f-OWL DL and f-OWL Lite to f-SHOIN(\mathbf{D}) and f-SHIF(\mathbf{D}), respectively, preserves satisfiability. That is, an f-OWL DL (resp. f-OWL Lite) axiom or fact is satisfied by a fuzzy interpretation \mathcal{I} if and only if the translation is satisfied by \mathcal{I} .⁸*

The above theorem can be shown by a simple recursive argument over the semantics of f-OWL and f-SHOIN(\mathbf{D}). It also shows that if O_1 and O_2 are two fuzzy ontologies and K_1 and K_2 are the fuzzy knowledge bases that result by applying the reduction technique to O_1 and O_2 , respectively, then $O_1 \models O_2$ iff $K_1 \models K_2$.

5.2. From KB entailment to KB satisfiability

Subsequently, it is easy to see that $K_1 \models K_2$ iff $K_1 \models A$ for each axiom A of K_2 . Hence, as a second step of the reduction we have to reduce f-SHOIN(\mathbf{D}) knowledge base entailment to f-SHOIN(\mathbf{D}) unsatisfiability. More precisely, we have to define a translation \mathcal{G} such that $K \models A$ iff $K \cup \{\mathcal{G}(A)\}$ is unsatisfiable. The definition of \mathcal{G} is depicted in Table 9.

⁸Note that we abuse the syntax using \mathcal{I} to represent both a fuzzy DL and an fuzzy OWL interpretation.

There are some remarks regarding the definition of \mathcal{G} . The reduction of fuzzy concept assertions and concept subsumptions have already been shown in [13]. On the other hand, since here we have to reduce the entailment of knowledge bases we had to consider the reduction of several non-standard DL axioms to KB satisfiability, like role subsumption, transitive role axioms and fuzzy role assertions. Firstly, the reduction of role subsumptions and transitive role axioms is a result of viewing these axioms as the two equisatisfiable subsumptions of the form $\exists R.\{y\} \sqsubseteq \exists S.\{y\}$ and $\exists R.\exists R.\{y\} \sqsubseteq \exists R.\{y\}$, respectively. Secondly, observe that we give several reductions for the entailment of fuzzy role assertions $((a, b) : R) \triangleright n$. The reduction in classical OWL follows the use of value restrictions. More precisely, $\Sigma^c \models (a, b) : R$ iff $\Sigma^c \cup \{a : \forall R. \neg B, b : B\}$ is unsatisfiable for some b not appearing in Σ^c . We have tried to extend this result to also use value restrictions in f-DLs. As a consequence we realized that the reduction should distinguish between f-OWL that uses R - or S - implications for interpreting value restrictions. On the other hand it seems that it is still possible to provide a reduction regardless of the fuzzy operators used, by using existential restrictions as in the third alternative.

Theorem 5.2. *Let Σ and Σ' be f-SHOIN(\mathbf{D}) knowledge bases. Then $\Sigma \models \Sigma'$ iff the f-SHOIN(\mathbf{D}) knowledge base $\Sigma \cup \mathcal{G}(A)$ is unsatisfiable for every axiom A in Σ' .*

Proof: The proof is given in the appendix ■

Regarding the reduction of f-SHIF(\mathbf{D}) entailment to f-SHIF(\mathbf{D}) satisfiability a number of issues have to be taken under consideration. More precisely, the f-SHIF(\mathbf{D}) language does not support nominals, thus, the reduction method for role subsumption and transitive role axioms, presented in Table 9, cannot be used. This is also true in the case of crisp OWL Lite [4]. For that purpose a new transformation method has to be devised. Based on the translation method presented in [4] we can replace each nominal concept in Table 9 with a new atomic concept B not present in the KB. The new mapping that uses this new notion will be denoted by \mathcal{G}' . This yields a translation method for f-SHIF(\mathbf{D}).

Theorem 5.3. *Let Σ and Σ' be f-SHIF(\mathbf{D}) knowledge bases derived from OWL Lite ontologies. Then $\Sigma \models \Sigma'$ iff the f-SHIF(\mathbf{D}) knowledge base $\Sigma \cup \{\mathcal{G}'(A)\}$ is unsatisfiable, for every axiom $A \in \Sigma'$.*

Proof: The proof is given in the appendix ■

Please note that, the approach taken in [4] for the reduction of SHIF(\mathbf{D}) entailment to SHIF(\mathbf{D}) satisfiability is different than the approach taken here. More precisely, in [4], the entailment of an axioms $R \sqsubseteq S$ was reduced to the unsatisfiability of the concept $B \sqcap \exists R. (\forall S^-. \neg B)$, while the entailment of a transitivity axiom $\text{Trans}(R)$, was reduced to the unsatisfiability of the concept, $B \sqcap \exists R. (\exists R. (\forall R^-. \neg B))$ where, in both cases, B is a new concept not present in the KB. In the case of f-SHIF(\mathbf{D}), a similar reduction is not possible.

As we can easily note from the above, in the entailment of concept and role subsumption and transitive role axioms one has to check for the satisfiability of the ABoxes for all degrees $n \in [0, 1]$. As we have already discussed Straccia [13] has provided a practical reduction for checking concept subsumption in $f_{KD}\text{-}\mathcal{ALC}$, where it suffices to check for the satisfiability of two specific ABoxes for two arbitrary degrees $n_1 \in (0, 0.5]$ and $n_2 \in (0.5, 1]$. It is obvious important to know whether this result can be extended to to fuzzy DLs with arbitrary operators. Unfortunately, this result *cannot* be generalized.

Example 5.4 (counterexample). *Let the KB Σ consisting just of the TBox $\mathcal{T} = \{C \sqsubseteq F, D \sqcup \neg D \sqsubseteq E\}$ and lets check whether $\Sigma \models C \sqsubseteq E$ by using this method. Obviously, the subsumption does not hold. Suppose that we consider the probabilistic sum and the Lukasiewicz negation as fuzzy operators. In every model \mathcal{I} of Σ , we should have $(D \sqcup \neg D)^{\mathcal{I}}(a) \leq E^{\mathcal{I}}(a) \Rightarrow D^{\mathcal{I}}(a) + 1 - D^{\mathcal{I}}(a) - (1 - D^{\mathcal{I}}(a)) \cdot D^{\mathcal{I}}(a) \leq E^{\mathcal{I}}(a) \Rightarrow 1 - D^{\mathcal{I}}(a) + (D^{\mathcal{I}}(a))^2 \leq E^{\mathcal{I}}(a)$. The function of the left-hand side is minimized at 0.5 and the minimum is $1 - 0.5 + 0.5^2 = 0.75$, consequently $E^{\mathcal{I}}(a) \geq 0.75$ for any $a \in \Delta^{\mathcal{I}}$. Hence, if we consider the degrees $n_1 = 0.4$ and $n_2 = 0.6$ then obviously $(b : E) < n$ cannot be satisfied in any model \mathcal{I} of Σ for any $n \in \{n_1, n_2\}$ thus $\Sigma \cup \{(b : C) \geq n, (b : E) < n\}$.*

The difference with f_{KD} -DLs is that although the minimum and the maximum of the semantic functions is again at 0.5, the value of the functions do not increase (or decrease) due to the idempotency of min and max. Nevertheless, even in the case of $f_{KD}\text{-}\mathcal{SHOIN}(\mathbf{D})$ the result of Straccia does not generalize straightforwardly. This is because as it is well-known in the presence of nominals, the ABox should be taken into consideration when testing for subsumption. In our case the ABox contains fuzzy assertions, thus these degrees are also expected to be important in the check for subsumption. The following, (counter)example makes this case explicit.

Example 5.5 (counterexample). *Let the KB $\Sigma = \langle \{\top \sqsubseteq \{a\}\}, \emptyset, \{a : D = 0.8, a : C = 0.4\} \rangle$. We want to use the practical reduction from [13] to check whether $\Sigma \models C \sqsubseteq D$. The method sais that we have to check the unsatisfiability of $\Sigma' = \Sigma \cup \{(b : D) \geq n, (b : C) < n\}$ for two $n \in \{n_1, n_2\}$ for some new b . If we select the values $n_1 = 0.2$ and $n_2 = 0.9$, then both times Σ' is unsatisfiable. The TBox axiom forces b to be identified with a (since a is the only object in $\Delta^{\mathcal{I}}$, then $a^{\mathcal{I}} = b^{\mathcal{I}}$), thus we actually have $(a : D) \geq n$ and $(a : C) < n$ and for n_1 the second assertion is unsatisfiable, while for n_2 the first one.*

Consequently, in the presence of nominals and ABoxes we have the following:

Theorem 5.6. *Let C and D be two $f_{KD}\text{-}\mathcal{SHOIN}(\mathbf{D})$ -concepts and let $\Sigma = \langle \mathcal{T}, \mathcal{R}, \mathcal{A} \rangle$ be an knowledge base in this language. Then, $\Sigma \models C \sqsubseteq D$ iff $\langle \mathcal{T}, \mathcal{R}, \mathcal{A} \cup \{(a : C) \geq n, (a : D) < n\} \rangle$ is unsatisfiable for each $n \in \{n', 1 - n' \mid (\alpha \bowtie n') \in \mathcal{A}, \text{ where } \alpha \in \{a : C, (a, b) : R\}\}, \cup \{0, 0.5, 1\}$.*

For the rest of fuzzy DLs, in order to provide practical reasoning, one can formalize the reasoning problem as an optimisation problem [10, 39] under certain constraints in order to determine the solvability or unsolvability of the system and finally the entailment or non-entailment of the axiom. The reader is referred to [39] for more information on this reduction.

The reduction of OWL entailment to $f\text{-}\mathcal{SHOIN}(\mathbf{D})$ satisfiability, presented in this section, together with the recent results on reasoning with very expressive fuzzy DLs [14, 16] and with general inclusion axioms [15], implies that at the current moment we can fully support reasoning for the f_{KD} -OWL DL ontology language, i.e. for fuzzy OWL that uses the Lukasiewicz complement, the Gödel t-norm and t-conorm and the Kleene-Dienes fuzzy implication. A reasoning algorithm for a slightly less expressive fragment of f_{KD} -OWL, i.e. the reasoning algorithm for in the FiRE fuzzy reasoning engine [40] and some results regarding its usefulness in multimedia analysis tasks [5] and ontology mapping validation [7] have been investigated.

6. Extending the Simplistic f-OWL Extension

6.1. Fuzzy Nominals and fuzzy one-of

Fuzzy nominals and the fuzzy one-of constructor were first introduced by Bobillo et al. [24]. As the authors note, with such a constructor one can create fuzzy concepts by enumerating their members together with their degrees of membership in an analogous way as one can enumerate the elements of a crisp set in classical DLs with the one-of constructor. For example, we can describe the concept of German speaking countries as:

$$\text{GermanSpeaking} \equiv \{(germ, 1), (aus, 1), (switz, 0.67)\}.$$

More formally, if $o, o_1, \dots, o_k \in \mathbf{I}_A$, $c, c_1, \dots, c_k \in \mathbf{I}_D$, $n, n_1, \dots, n_k \in (0, 1]$ and $p, k \in \mathbb{N}$, then also the following are $f\text{-}\mathcal{SHOIN}(\mathbf{D})$ -concepts:

$$\begin{aligned} &\{o, n\}, \{(o_1, n_1), \dots, (o_k, n_k)\} \\ &\{c, n\}, \{(c_1, n_1), \dots, (c_k, n_k)\} \end{aligned}$$

Table 10 summarizes the semantics of the new constructors. The semantics of fuzzy nominals result easily by considering the semantics of fuzzy one-of [24] if we consider $n = 1$. In [16], in order to distinguish this fuzzy $\mathcal{SHOIN}(\mathbf{D})$ extension with the we called the language $f\text{-}\mathcal{SHO}_f\mathcal{IN}(\mathbf{D})$.

By using $f\text{-}\mathcal{SHO}_f\mathcal{IN}(\mathbf{D})$ as a logical base one can extend our simplistic extension of f-OWL to also allow for fuzzy nominals and the one-of constructor. Table 11 summarizes the abstract syntax, respective fuzzy DL syntax and semantics of the relevant axioms.

As one can note we have not included a fuzzy extension of hasValue restrictions. We argue that it is currently not very clear if such axioms should use fuzzy nominals. As we

Table 10: Syntax and Semantics of f- $\mathcal{SHO}_f\mathcal{IN}(\mathbf{D})$ -concepts

Constructor	DL Syntax	Semantics
nominal	$\{o, n\}$	$\{o, n\}^{\mathcal{I}}(a) = \begin{cases} n, & a = o^{\mathcal{I}} \\ 0, & \text{otherwise} \end{cases}$
individual one-of	$\{(o_i, n_i)\}$	$\{(o_1, n_1), \dots, (o_k, n_k)\}^{\mathcal{I}}(a) = \sup_{a=o_i^{\mathcal{I}}, 1 \leq i \leq k} n_i$
datatype nominal	$\{c, n\}$	$\{c, n\}^{\mathcal{I}}(y) = \begin{cases} n, & y = c^{\mathbf{D}} \\ 0, & \text{otherwise} \end{cases}$
datatype one-of	$\{(c_i, n_i)\}$	$\{(c_1, n_1), \dots, (c_k, n_k)\}^{\mathcal{I}}(y) = \sup_{y=c_i^{\mathbf{D}}, 1 \leq i \leq k} n_i$

Table 11: Class descriptions and axioms of f-OWL with fuzzy one-of

Abstract Syntax	DL Syntax	Semantics
$\text{oneOf}((o_1, n_1) (o_2, n_2) \dots)$	$\{(o_i, n_i)\}$	$\{(o_i, n_i)\}^{\mathcal{I}}(a) = \sup_{a=o_i^{\mathcal{I}}, 1 \leq i \leq k} n_i$
$\text{oneOf}((c_1, n_1) (c_2, n_2) \dots)$	$\{(c_i, n_i)\}$	$\{(c_i, n_i)\}^{\mathcal{I}}(y) = \sup_{y=c_i^{\mathbf{D}}, 1 \leq i \leq k} n_i$
$\text{EnumeratedClass}(A (o_1, n_1) \dots (o_k, n_k))$	$A \equiv \{\dots, (o_i, n_i), \dots\}$	$A^{\mathcal{I}}(a) = \sup_{a=o_i^{\mathcal{I}}, 1 \leq i \leq k} n_i$

have already seen this OWL constructor originates from the *fills* constructor. Intuitively, an axiom of the form $A \sqsubseteq R : o$ means that every object $a \in A^{\mathcal{I}}$ is also connected with the specific object $o^{\mathcal{I}}$ through $R^{\mathcal{I}}$. Stated otherwise $o^{\mathcal{I}}$ fills $R^{\mathcal{I}}$ for every $a \in A^{\mathcal{I}}$. In the presence of existential restrictions this constructor becomes a syntactic sugar and the axioms can be written as $A \sqsubseteq \exists R.\{o\}$. In fuzzy DLs the concept $R : o$ is a fuzzy set with membership function $(R : o)^{\mathcal{I}}(d) = R^{\mathcal{I}}(d, o^{\mathcal{I}})$ (as we have defined in Table 4), which is quite different from the semantics that result by a concept of the form $\exists R.\{o, n\}$ with membership function $\sup_c t(R^{\mathcal{I}}(d, c), \{o, n\}^{\mathcal{I}}(c)) = t(R^{\mathcal{I}}(d, c), n)$.

In the following we provide an investigation about the syntactic sugar constructors of fuzzy one-of/enumeration, similarly as we have done in previous sections about for example domain and range restrictions.

As we have shown before (Section 4) the one-of constructor becomes a syntactic sugar in the presence of nominals and disjunctions even in the simplistic extension of f-OWL. Now in the case of fuzzy nominals and fuzzy one-of, things are slightly more complicated. Actually, we can distinguish the following cases:

1. If our f-OWL language uses the Gödel t -conorm for interpreting conjunctions, then the fuzzy one-of operator can be expressed in terms of fuzzy nominals and disjunction, since obviously, k is finite and thus sup is actually max.
2. If we enforce the *Unique Names Assumption (UNA)*, i.e. that each individual in the ABox and each nominal represents a different object in the domain of interpretation (an assumption that we remark is mainly *not* assumed in DLs), then again it is the

case that:

$$\{(o_1, n_1), \dots, (o_k, n_k)\} \equiv \{o_1, n_1\} \sqcup \dots \sqcup \{o_k, n_k\}$$

regardless of the t -conorm used. This is because for each $a \in \Delta^{\mathcal{I}}$, a would be equivalent to only one of $o_i^{\mathcal{I}}$, say with $o_\ell^{\mathcal{I}}$, thus $\{o_i, n_i\}^{\mathcal{I}}(a) = 0$ for $1 \leq i \neq \ell \leq k$ and $\{o_\ell, n_\ell\}^{\mathcal{I}}(a) = n_\ell$. Hence, $\{(o_1, n_1), \dots, (o_k, n_k)\}^{\mathcal{I}}(a) = \max(0, \dots, n_\ell, \dots, 0) = n_\ell$, while on the other hand due to the boundary conditions of t -conorms, $(\{o_1, n_1\} \sqcup \dots \sqcup \{o_k, n_k\})^{\mathcal{I}}(a) = u(0, \dots, n_\ell, \dots, 0) = n_\ell$.

3. In case we do not have UNA and we use superidempotent t -conorms, then the two forms do not coincide. Needless to say the semantics that result by fuzzy nominals and disjunctions may result to strange effects, as e.g. if $a = o_1^{\mathcal{I}} = o_2^{\mathcal{I}}$, then $(\{o_1, n_1\} \sqcup \{o_2, n_2\})^{\mathcal{I}}(a) = u(n_1, n_2) > n_1, n_2$, i.e. although we are actually referring to one object of the domain its degree in the concept strictly increases.

In a similar way as above in order to represent such concepts in f-OWL one has to extend its abstract and concrete RDF/XML syntax. More precisely, the abstract syntax can be extended to the following:

description	::=	'oneOf(' { individualURI [degree] } ')
dataRange	::=	'oneOf(' { dataLiteral [degree] } ')
axiom	::=	'EnumeratedClass(' classID ['Deprecated'] { annotation } { individualID [degree] } ')

while for example, the concept of German speaking countries could be represented in RDF/XML as follows:

```
<owl:Class rdf:ID="GermanSpeaking">
  <owl:oneOf rdf:parseType="Collection">
    <Country rdf:about="#Germany"/>
    <Country rdf:about="#Austria"/>
    <Country rdf:about="#Switzerland" owlx:degree="0.67"/>
  </owl:oneOf>
</owl:Class>
```

The mapping of the new features of f-OWL into fuzzy DLs is again quite straightforward (following the same principles as mapping the simplistic f-OWL to $f\text{-}\mathcal{SHOIN}(\mathbf{D})$) if we use $f\text{-}\mathcal{SHO}_f\mathcal{IN}(\mathbf{D})$ as the underlying fuzzy DL. Nevertheless, it is important to investigate whether a practical reduction for checking subsumption of concepts that potentially involve fuzzy nominals exists in $f_{KD}\text{-}\mathcal{SHO}_f\mathcal{IN}(\mathbf{D})$. As we have already seen a practical reduction in the presence of nominals is already “problematic” since the degrees that appear in the ABox are expected to be important in the check for subsumption. Here, degrees can even appear in concepts due to the presence of fuzzy nominals. Thus, in this case even in the

absence of an ABox, one has to reason over the degrees that possibly appear in the fuzzy nominals. Let for example, $\emptyset \models \{o, 0.6\} \sqsubseteq \{o, 0.4\}$. Clearly this entailment does not hold, nevertheless in order not to report a false positive subsumption, the degrees considered in the reduced KB $\{a : \{o, 0.6\} \geq n, a : \{o, 0.4\} < n\}$ should be the degrees $n_1 = 0.6$ and $n_2 = 0.4$ of which the first choice leads to a satisfiable ABox thus correctly identifying the non-subsumption.

Nevertheless, there is still some problems with this approach. Consider, for example the subsumption check $\Sigma \models C \sqcup \{o, 0.4\} \sqsubseteq \{o, 0.8\}$. This inference obviously does not hold but with the above approach we would falsely report that it does, since $\{(a : C \sqcup \{o, 0.4\}) \geq n, (a : \{o, 0.8\}) < n\}$ is unsatisfiable for both $n \in \{0.4, 0.8\}$. The issue here is that we do not take correctly into account the semantics of the right-hand side nominal. More precisely, observe that for any fuzzy nominal $\{o, n_2\}$ in the right-hand side of a subsumption, the assertion produced by the reduction, $(a : \{o, n_2\}) < n_2$, is always trivially unsatisfiable. To remedy this effect instead of 0.8, we could use the degree $0.8 + \epsilon$, where ϵ is a small number that can be calculated easily by ordering the degrees of every fuzzy nominal (together with the degrees $\{0, 0.5, 1\}$) and by taking the half of the smallest difference. Then, $\{(a : C \sqcup \{o, 0.4\}) \geq 0.8 + \epsilon, (a : \{o, 0.8\}) < 0.8 + \epsilon\}$ is satisfiable.

Corollary 6.1. *Let C and D be two $f_{KD}\text{-SHO}_f\text{IN}(\mathbf{D})$ -concepts and let $\Sigma = \langle \mathcal{T}, \mathcal{R}, \mathcal{A} \rangle$ be an knowledge base in this language. Then, $\Sigma \models C \sqsubseteq D$ iff $\langle \mathcal{T}, \mathcal{R}, \mathcal{A} \cup \{(a : C) \geq n, (a : D) < n\} \rangle$ is unsatisfiable for each $n \in X^\Sigma$ where X^Σ is defined as follows:*

$$\begin{aligned} X^\Sigma &= \{0, 0.5, 1\} \cup \\ &\quad \{n, 1 - n \mid (\alpha \bowtie n) \in \mathcal{A}, \text{ where } \alpha \in \{a : C, (a, b) : R\} \cup \\ &\quad \{n_i \mid \text{for every } \{\dots, (o_i, n_i), \dots\} \text{ appearing in } \mathcal{T}, \mathcal{A} \text{ or } D\} \cup \\ &\quad \{n_i + \epsilon \mid \text{for every } \{\dots, (o_i, n_i), \dots\} \text{ appearing in } D\} \end{aligned}$$

6.2. Fuzzy Subsumption Axioms

Straccia [18] proposed the extension of concept and role inclusion axioms defining what is called fuzzy subsumption axioms. If C, D are $f\text{-SHOIN}(\mathbf{D})$ concepts and $n \in (0, 1]$, then $\langle C \sqsubseteq D, n \rangle$ is a fuzzy subsumption axiom; similarly with roles. Intuitively, these axioms say that the degree of subethood of C to D is at-least equal to n , thus allowing a form of fuzzy subethood. The semantics of such axioms are provided by viewing an inclusion axiom $C \sqsubseteq D$ as a First-Order formula $\forall x. C(x) \rightarrow D(x)$ and interpreting \forall as inf and \rightarrow with a fuzzy implication. In other words we say that a fuzzy interpretation \mathcal{I} satisfies $\langle C \sqsubseteq D, n \rangle$ iff $\inf_{a \in \Delta^{\mathcal{I}}} \mathcal{J}(C^{\mathcal{I}}(a), D^{\mathcal{I}}(a)) \geq n$.

Several authors [41, 42, 43] have tried to axiomatize the notion of fuzzy subsumption in the Fuzzy Set literature. Most of them give different semantics to fuzzy inclusions, but it seems that fuzzy implications, and more particularly R -implications, are somehow the intersection of all the above approaches. On the other hand S -implications might lead to counterintuitive results, as noted in [24]. For example, if we consider the Kleene-Dienes fuzzy implication then $\inf_{a \in \Delta^{\mathcal{I}}} \mathcal{J}(C^{\mathcal{I}}(a), C^{\mathcal{I}}(a)) = \max(1 - C^{\mathcal{I}}(a), C^{\mathcal{I}}(a)) = 0.5$,

which means that a concept C only halfly subsumes itself. Moreover, an axiom of the form $\langle \text{TallPerson} \sqsubseteq \text{Human}, 1 \rangle$, which states that every tall person is a human (and obviously no degree in the subsumption is required) forces TallPerson and Human to be interpreted as crisp concepts, since $\max(1 - \text{TallPerson}^{\mathcal{I}}(a), \text{Human}^{\mathcal{I}}(a)) = 1$ iff $1 - \text{TallPerson}^{\mathcal{I}}(a) = 1$ or $\text{Human}^{\mathcal{I}}(a) = 1$, although TallPerson is obviously a fuzzy concept. This would not be a problem with R -implications since $\mathcal{J}(a, b) = 1$ iff $a \leq b$, i.e. in this case $\text{TallPerson}^{\mathcal{I}}(a) \leq \text{Human}^{\mathcal{I}}(a)$, which gives the semantics of the standard subsumption we defined in Section 3. Nevertheless, even the use of R -implications can lead to undesired situations. More precisely, R -implications are not very fine grained in the following sense: Consider three concepts C, D and E with $C^{\mathcal{I}}(a_i) = 0.3$, $D^{\mathcal{I}}(a_i) = 0.9$, for $1 \leq i \leq 10$, $C^{\mathcal{I}}(a_{11}) = 0.3$, $D^{\mathcal{I}}(a_{11}) = 0.2$ and $E^{\mathcal{I}}(a_j) = 0.2$ for $1 \leq j \leq 11$. Then $\inf_b \mathcal{J}(C^{\mathcal{I}}(b), D^{\mathcal{I}}(b)) = \mathcal{J}(0.3, 0.2) = \inf_b \mathcal{J}(C^{\mathcal{I}}(b), E^{\mathcal{I}}(b))$, i.e. although all but just one object of $\Delta^{\mathcal{I}}$ have a significantly greater membership degree in $D^{\mathcal{I}}$ compared to $C^{\mathcal{I}}$ the degree of subsethood of C to D is the same as that of C to E , even though in this case all objects belong to $E^{\mathcal{I}}$ to a less degree than they belong to $C^{\mathcal{I}}$. Concluding, we note that although most works in [41, 42, 43] do allow for the use of *some* R -implications for interpreting fuzzy inclusion all of them usually advocate for the use of other types of fuzzy operators, and more precisely for fuzzy aggregation type operators [20] which provide a more fine grained approach. Unfortunately, we currently don't know how to reason with fuzzy inclusion axioms that are defined by such operators.

Now from a practical point of view it is also not very clear how fuzzy subsumption axioms could or should be used in practice. It is definitely almost impossible to expect from average users that have basic understanding of OWL constructors to start writing fuzzy subsumption axioms (even if such a functionality was provided by tools). This would require deep understanding of the semantics and the different properties of fuzzy implications, as well as the consequences that such an axiom would have to the knowledge base. For example, inconsistencies could arise very easily, as e.g. having $\langle C \sqsubseteq D, 0.6 \rangle \in \mathcal{T}$ and asserting $a : C = 0.8$, $a : D = 0.3$ (just test for the Lukasiewicz implication and for many more). The latter one is generally a problematic point since even fuzzy assertions are usually expected to result automatically by a fuzzy partitioning system [20] that would map e.g. height measurements in a database about persons into concepts such as TallPerson , MediumHightPerson , etc. rather than manually, so the chances of running into such situations is high. We also argue that these degrees have a more “statistical” meaning since they are applied over classes and result by looking over all members of a class rather than a specific assertion, so they are difficult to be asserted. This is also advocated by the work of Young [43] that connects fuzzy subsethood with a notion of entropy and probability. Nevertheless, these axioms could be understood quite naturally as (fuzzy) mappings between concepts. More precisely, in [7] we have applied such axioms on the domain of ontology mapping/alignment to formalize mappings between different but overlapping ontologies and perform mapping validation. For example, if \mathcal{O}_1 and \mathcal{O}_2 are two different (created by different persons) ontologies about university courses then a (semi)automatic mapping algorithm could identify the mapping $\langle \mathcal{O}_1 : \text{Msc_Thesis} \sqsubseteq$

$O_2 : \text{Master_Thesis}, 0.8$). Then based on this mapping we can translate (crisp) assertions $a : \text{Msc_Thesis}$ in O_1 into (fuzzy) assertions $a : \text{Master_Thesis} \geq 0.8$ in O_2 via the mapping and then use fuzzy reasoners to perform mapping validation [7]. A similar treatment of fuzzy inclusions as fuzzy mappings (but for a different application domain) was also adopted in [6].

We could potentially extend f-OWL with the ability to state fuzzy subsumptions. Nevertheless, since all OWL axioms are interpreted as DL axioms it is not clear where or if this extension should at some point stop. More precisely, besides extending the `subClassOf`, `complete` and `partial` axioms, one could also even extend domain, range, functionality and disjointness axioms since all these are translated into DL inclusion axioms. Nevertheless, the extension of the most relevant OWL constructors as well as the extension of the abstract syntax are shown in Table 12, where `modality` represents either the keyword `partial` (defining subsumption) or `complete` (defining equivalence) and `description` is an OWL class description.

Table 12: f-OWL extensions with fuzzy subsumption

Constructor	DL Syntax	Semantics
<code>Class(A partial $C_1 \dots C_k n$)</code>	$\langle A \sqsubseteq \prod_{i=1}^k C_i, n \rangle$	$\inf_a \mathcal{J}(A^{\mathcal{I}}(a), \prod_{i=1}^k C_i^{\mathcal{I}}(a)) \geq n$
<code>Class(A complete $C_1 \dots C_k n$)</code>	$\langle A \equiv \prod_{i=1}^k C_i, n \rangle$	$\inf_a \mathcal{J}(A^{\mathcal{I}}(a), \prod_{i=1}^k C_i^{\mathcal{I}}(a)) = n$
<code>SubClassOf(C_1, C_2)</code>	$\langle C_1 \sqsubseteq C_2, n \rangle$	$\inf_a \mathcal{J}(C_1^{\mathcal{I}}(a), C_2^{\mathcal{I}}(a)) \geq n$
<code>axiom ::= 'Class(' classID modality { annotation } { description } [degree] ')</code>		
<code>axiom ::= 'SubClassOf(' description description [degree] ')</code>		

The translation of the extended f-OWL axioms into f- $\mathcal{SHOIN}(\mathbf{D})$ is quite straightforward if we consider f- $\mathcal{SHOIN}(\mathbf{D})$ with fuzzy subsumption axioms as the underlying fuzzy DL. On the other hand for the reduction of KB entailment to unsatisfiability we can distinguish two cases. If we define fuzzy subsumption with the aid of S -implications, then due to their definition, $\langle C \sqsubseteq D, n \rangle$ can be seen as $(a : \neg C \sqcup D) \geq n$, for every a . Thus, the entailment of a concept subsumption can be seen as an entailment of a (simple) fuzzy assertion.

Corollary 6.2. *Let C and D be two f_S - $\mathcal{SHOIN}(\mathbf{D})$ -concepts and Σ an f_S - $\mathcal{SHOIN}(\mathbf{D})$ KB. Then, $\Sigma \models \langle C \sqsubseteq D, n \rangle$ iff $\Sigma' = \Sigma \cup \{(a : \neg C \sqcup D) < n\}$ is unsatisfiable for some new a not in Σ .*

Observe that this is also a practical reduction, since we only need to check the unsatisfiability of Σ' for just one degree n , which is in turn even more optimal than the simplistic

case of $f_{KD}\text{-SHOIN}(\mathbf{D})$, where we need to check in the best case for two degrees⁹. This is again a result of defining fuzzy subsumption with the aid of S -implications, which actually gives them a more concepts-like meaning. We can parallelize this with crisp DLs where checking for $C \sqsubseteq D$ is reduced into unsatisfiability of $a : \neg(\neg C \sqcup D)$ for some new a .

For the case of R -implications we have the following: A fuzzy interpretation \mathcal{I} satisfies $\langle C \sqsubseteq D, n \rangle$ if $\inf_a \mathcal{J}(C^{\mathcal{I}}(a), D^{\mathcal{I}}(a)) \geq n$. Since this holds for inf it would hold for any arbitrary b , i.e. $\mathcal{J}(C^{\mathcal{I}}(b), D^{\mathcal{I}}(b)) \geq n \Rightarrow t(C^{\mathcal{I}}(b), n) \leq D^{\mathcal{I}}(b)$. The latter inequation resembles the semantics of the classical fuzzy subsumption. Thus, we can provide the following:

Corollary 6.3. *Let C and D be two $f_R\text{-SHOIN}(\mathbf{D})$ -concepts and Σ an $f_R\text{-SHOIN}(\mathbf{D})$ KB. Then, $\Sigma \models \langle C \sqsubseteq D, n \rangle$ iff $\Sigma \cup \{(b : C) \geq m_1, (b : D) < m\}$ under $t(m_1, n) = m$ for every $m \in [0, 1]$.*

This is actually a generalization of the reduction shown in Table 9, since for the simplistic case $n = 1$, thus, $t(m_1, 1) = m_1 = m$ which gives the normal reduction. Nevertheless, still in a practical setting (i.e. in a reasoning algorithm) one has to formalize the problem as e.g. in [39].

At this point we would like to conclude our investigations over fuzzy extensions of OWL. As we have remarked several times one might argue that there are more (fuzzy) features proposed for fuzzy DL languages in the literature which are missing from our presentation. Some examples are concept modifiers [12], fuzzy quantifiers [21] and comparison expressions [22]. As we have emphasized from the beginning our intention is neither to perform an overview of all these features and provide an overall fuzzy-OWL extension covering everything that exists in the literature nor to advocate in favour or against any of these that we have not included here. Our goal is to investigate various points that have not been addressed before regarding fuzzy extensions of OWL, even in the very simplistic fuzzy extension that we presented in Section 4. These include the semantics of syntactic sugar axioms as well as the reduction of f-OWL entailment to that of f-DL satisfiability. This analysis shows that careful selection of operators and mapping to expressive fuzzy DLs is needed in order to provide the intended semantics. Subsequently, we have extended our investigations to two popular features from the fuzzy DL literature and more precisely to fuzzy nominals and fuzzy one-of. We have seen that even for such seemingly harmless or natural extensions several issues arise, when it comes to the mapping of fuzzy enumerations into fuzzy DL axioms or the (practical) reduction of entailment to unsatisfiability, even for fuzzy DLs such as f_{KD} -DLs which are known to have quite similar properties with classical DLs regarding reasoning mechanisms.

⁹Recall that if nominals and an ABox are present then we need to check for far more degrees

7. Conclusions

Imprecise and vague knowledge is apparent in many real life applications and domains. Some examples are multimedia analysis [5], Semantic Portals [6], ontology mapping [7], Semantic Web Services matching [8] and many more. Representing and reasoning with such type of information is expected to play a significant role in assisting these applications provide better results. To this extent we have investigated fuzzy extensions of the OWL web ontology language, creating fuzzy OWL (f-OWL). This extension builds upon previous results that have been achieved in the field of fuzzy Description Logics [10, 12, 11, 23, 24]. Firstly, we have presented a simplistic fuzzy extension of OWL which is based only on fuzzy instance relations. For this extension we have presented the semantics, abstract syntax and an RDF/XML syntax of fuzzy OWL. Moreover, we have investigated properties of the semantics, like the connection between crisp and fuzzy interpretations, as well as the semantics and properties of syntactic sugar axioms of f-OWL like concept disjointness, property range axioms, functional role axioms and the one-of/enumeration constructor. Finally, we have presented a transformation technique that reduces the problem of f-OWL ontology entailment to the problem of $f\text{-}SHOIN(\mathbf{D})$ knowledge base satisfiability. To this extend we have investigated the reduction of the entailment of several $f\text{-}SHOIN(\mathbf{D})$ axioms into KB satisfiability which have not been considered before. Consequently, this reduction implies that we can provide reasoning support over f-OWL ontologies by applying reasoning over the reduced $f\text{-}SHOIN(\mathbf{D})$ knowledge bases. Subsequently, we have extended the simplistic f-OWL with two features and more precisely with fuzzy enumerations and fuzzy subsumption axioms. For both these features we investigate their semantics, we show the extended abstract syntax and a possible RDF/XML syntax, and finally the reduction to f-DL KB satisfiability.

Nevertheless, much work needs to be done until we provide full support for handling and managing vague information in Semantic Web applications. First of all there needs to be support of the new features of f-OWL as well as other fuzzy features from specialized editing tools. This will help the easy and rapid development of fuzzy knowledge bases, which would lead to wider acceptance from the research community. This is actually very difficult at the current point since no f-OWL standard extension exists. Additionally, there is a number of open research problems related to fuzzy DLs. More precisely, implementing and optimizing fuzzy Description Logic reasoners is also a very important issue. Moreover, the development of tableaux reasoning algorithms for expressive fuzzy DL systems other than f_{KD} -DLs is also another open research problem. Finally, we also note that in the current work we have not provided any treatment of a fuzzy extension of the upcoming OWL2 w3C standard (an extension of OWL). A first account of fuzzy OWL2 can be found in [44] as well as a fuzzy extension of the $SRIOQ$ language (the underlying DL of OWL2) in [45].

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A. Proofs of Theorems and Lemmas

Proof of Theorem 3.1. The proof of this theorem can be shown by applying induction on the structure of concepts and showing that if $C^{\mathcal{I}}(a) = 1$, then $a \in C^{\mathcal{I}}$. Most of the cases follow easily if we consider the boundary conditions of the fuzzy operators and the fact that we are restricted only to degrees zero and one. More precisely, if $a, b \in \{0, 1\}$, then $t(a, b) = 1$ iff $a = b = 1$, $u(a, b) = 1$ iff either $a = 1$ or $b = 1$, $\mathcal{J}(a, b) = 1$ iff either $a = 0$ or $b = 1$ for both S and R -implications and $c(a) = 1$ iff $a = 0$, giving the semantics of the classical operators. Moreover, from $A^{\mathcal{I}}(a) = 1$, where A is an atomic concept and $R^{\mathcal{I}}(a, b) = 1$ it obviously follows that $a \in A^{\mathcal{I}}$ and $\langle a, b \rangle \in R^{\mathcal{I}}$. Similarly, $A^{\mathcal{I}}(a) = 0$ in classical notion is $a \notin A$. Finally, for inverse roles due to semantics $R^{\mathcal{I}}(a, b) = (R^-)^{\mathcal{I}}(b, a)$. Thus if $R^{\mathcal{I}}(a, b) = 1 = (R^-)^{\mathcal{I}}(b, a)$, then $\langle a, b \rangle \in R^{\mathcal{I}}$ and also $\langle b, a \rangle \in (R^-)^{\mathcal{I}}$. \square

Proof of Lemma 4.1. Suppose that $C \sqcap D \sqsubseteq \perp$, holds for all fuzzy interpretations \mathcal{I} . This means that $\forall x \in \Delta^{\mathcal{I}}. t(C^{\mathcal{I}}(x), D^{\mathcal{I}}(x)) \leq 0$. Since, the fuzzy triple satisfies the law of contradiction we have $\forall x \in \Delta^{\mathcal{I}}. t(C^{\mathcal{I}}(x), D^{\mathcal{I}}(x)) \leq t((\neg D)^{\mathcal{I}}(x), D^{\mathcal{I}}(x))$ and due to the monotonicity property of t-norms we get $\forall x \in \Delta^{\mathcal{I}}. C^{\mathcal{I}}(x) \leq \neg D^{\mathcal{I}}(x)$. Hence, we can abstract from interpretations and in general write that $C \sqsubseteq \neg D$, holds.

Now suppose that $C \sqsubseteq \neg D$, for all \mathcal{I} . This means that $\forall x \in \Delta^{\mathcal{I}}. C^{\mathcal{I}}(x) \leq (\neg D)^{\mathcal{I}}(x)$. Similarly, as above, we can get, $\forall x \in \Delta^{\mathcal{I}}. t(C^{\mathcal{I}}(x), D^{\mathcal{I}}(x)) \leq t((\neg D)^{\mathcal{I}}(x), D^{\mathcal{I}}(x))$, and finally, $\forall x \in \Delta^{\mathcal{I}}. t(C^{\mathcal{I}}(x), D^{\mathcal{I}}(x)) \leq 0$, for all \mathcal{I} . Again, in abstract notation we can simply write, $C \sqcap D \sqsubseteq \perp$. \square

Proof of Theorem 5.2. Given that $\Sigma \models \Sigma'$ iff $\Sigma \models A$ for each axiom A in Σ' we only need to show that $\Sigma \models A$ iff $\Sigma \cup \mathcal{G}(A)$ is unsatisfiable for any given axiom A . For that purpose we will examine each axiom presented in Table 9. In the following C, D are concepts, R, S are roles, a, b are individuals and x, y are fresh individuals.

- $\Sigma \models (a : C) \bowtie n$ iff $\Sigma \cup \{(a : C) \neg \bowtie n\}$ is unsatisfiable. We will only consider the case with $\bowtie = \geq$. All other cases can be shown similarly. If $\Sigma \models (a : C) \geq n$ then in every model of Σ it holds that $C^{\mathcal{I}}(a^{\mathcal{I}}) \geq n$, thus we can not find an interpretation where $C^{\mathcal{I}}(a) < n$, and as a result $\Sigma \cup \{(a : C) \neg \geq n\}$, with $\neg \geq \equiv <$, is unsatisfiable. For the converse, if \mathcal{I} is a model of Σ in which $C^{\mathcal{I}}(a) < n$, then \mathcal{I} also satisfies $(a : C) < n \equiv (a : C) \neg \geq n$ and thus, $\Sigma \cup \{(a : C) \neg \geq n\}$ is satisfiable.

- $\Sigma \models ((a, b) : R) \bowtie n$ iff $\Sigma \cup \{(a : \forall R. \neg B) + \bowtie u(c(n), c(n)), (b : B) \geq n\}$ is unsatisfiable when the fuzzy OWL language uses an S -implications, while $\Sigma \models ((a, b) : R) \bowtie n$ iff $(a : \forall R. \neg B) \geq 1, (b : \neg B) \neg \bowtie n$ is unsatisfiable when the fuzzy OWL uses an R -implication. Without loss of generality we will prove only the case where $\bowtie = \geq$. If $\Sigma \models ((a, b) : R) \geq n$, then in every model \mathcal{I} of Σ it will hold that $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq n$. Then we have two cases:

1. Let the family of f_S -OWL. Then from the above we obtain $c(R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}})) \leq c(n)$. In order for \mathcal{I} to be a model of the first ABox, \mathcal{I} should also satisfy $b : B \geq n$ i.e. $B^{\mathcal{I}}(b^{\mathcal{I}}) \geq n \Rightarrow (\neg B)^{\mathcal{I}}(b^{\mathcal{I}}) \leq c(n)$. Consequently,

$$\begin{aligned} u(c(R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}})), (\neg B)^{\mathcal{I}}(b^{\mathcal{I}})) &\leq u(c(n), c(n)) \Rightarrow \\ \inf_{b^{\mathcal{I}} \in \Delta^{\mathcal{I}}} u(c(R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}})), (\neg B)^{\mathcal{I}}(b^{\mathcal{I}})) &\leq u(c(n), c(n)) \Rightarrow \\ (\forall R. \neg B)^{\mathcal{I}}(a^{\mathcal{I}}) &\leq u(c(n), c(n)). \end{aligned}$$

Summing up there exists no interpretation \mathcal{I} that is a model of Σ , and it satisfies both $b : B \geq n$ and $(a : \forall R. \neg B) > u(c(n), c(n))$, ($+ \geq \Rightarrow$).

2. Let the family of f_R -OWL. In order for \mathcal{I} to be a model of the second ABox it should satisfy the assertion $(b : \neg B) < n$ and thus $(\neg B)^{\mathcal{I}}(b^{\mathcal{I}}) < n$. Since for R -implications $\mathcal{J}_R(n_1, n_2) = 1$ iff $n_1 \leq n_2$, then since $(\neg B)^{\mathcal{I}}(b^{\mathcal{I}}) < n \leq R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}})$ we have $\mathcal{J}(R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}), (\neg B)^{\mathcal{I}}(b^{\mathcal{I}})) < 1$, hence also \inf would be lower than 1. Concluding $b : \neg B < n$ and $a : \forall R. \neg B \geq 1$ cannot be satisfied simultaneously in models of Σ .
3. Similarly as above, \mathcal{I} should satisfy $(b : B) \geq 1$, i.e. $B^{\mathcal{I}}(b^{\mathcal{I}}) \geq 1$, thus $t(R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}, B^{\mathcal{I}}(b^{\mathcal{I}}))) \geq t(n, 1) = n$. Since b is an arbitrary object, this would also hold for the supremum. Concluding, there is no interpretation \mathcal{I} that can satisfy $(a : \exists R. B) < n$.

For the converse we proceed by reduction to absurd. Let \mathcal{I} be a model of Σ and that the ABoxes, for the respective cases of fuzzy implications, are unsatisfiable, but to the contrary let $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) < n$, i.e. $\Sigma \not\models ((a, b) : R) \geq n$. Now we have two cases:

1. First consider S -implications. From $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) < n$ we obtain $c(R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}})) > c(n)$. Now \mathcal{I} can be extended such that $B^{\mathcal{I}}(b^{\mathcal{I}}) = n$, thus it satisfies $(b : C) \geq n$, but additionally $(\neg B)^{\mathcal{I}}(b^{\mathcal{I}}) = c(n) \geq c(n)$. Moreover, for every other $w \in \Delta^{\mathcal{I}}$ we can set $(\neg B)^{\mathcal{I}}(w) = 1$. Note that this is possible since B is a new concept in the knowledge base. Consequently, $\max(c(R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}})), (\neg B)^{\mathcal{I}}(b^{\mathcal{I}})) > u(c(n), c(n))$, but it also holds that $\max(c(R^{\mathcal{I}}(a^{\mathcal{I}}, w)), (\neg B)^{\mathcal{I}}(w)) = 1 > u(c(n), c(n))$. Concluding, \mathcal{I} is a model of Σ which satisfies $\{(a : \forall R. \neg B) + \bowtie u(c(n), c(n)), (b : B) \geq n\}$, absurd.
2. Now consider R -implications. Then \mathcal{I} can be extended in order to satisfy $(b : B) > c(n)$ and more precisely to hold that $B^{\mathcal{I}}(b^{\mathcal{I}}) = n_1 > c(n) \Rightarrow (\neg B)^{\mathcal{I}}(b^{\mathcal{I}}) =$

$c(n_1) < n$. Additionally, we also set $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) = n_2 < n$ with $c(n_1) \geq n_2$, but also $(\neg B)^{\mathcal{I}}(w) = 1$ for every other $w \in \Delta^{\mathcal{I}}$. Then, $(\neg B)^{\mathcal{I}}(b^{\mathcal{I}}) \geq R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}})$, and $(\neg B)^{\mathcal{I}}(w) \geq R^{\mathcal{I}}(a^{\mathcal{I}}, w)$ for every w . Thus, $\inf_{z \in \Delta^{\mathcal{I}}} \mathcal{J}(R^{\mathcal{I}}(a^{\mathcal{I}}, z), (\neg B)^{\mathcal{I}}(z)) = 1$ and \mathcal{I} is a model of Σ that satisfies $(a : \forall R. \neg B) \geq 1, (b : B) + \bowtie c(n)$, absurd.

3. Similarly, as above.

- $\Sigma \models a \doteq b$ iff $\Sigma \cup \{a \neq b\}$ is not satisfiable. If $\Sigma \models a \doteq b$, then in every model \mathcal{I} of Σ , $a^{\mathcal{I}} = b^{\mathcal{I}}$, so \mathcal{I} cannot satisfy $a \neq b$. For the converse, if $\Sigma \cup \{a \neq b\}$ is not satisfiable, then in every model \mathcal{I} of Σ , $a^{\mathcal{I}} = b^{\mathcal{I}}$, so $\Sigma \models a \doteq b$.
- $\Sigma \models a \neq b$ iff $\Sigma \cup \{a = b\}$ is not satisfiable. This case can be show similarly to the previous one.
- $\Sigma \models C \sqsubseteq D$ iff $\Sigma' = \Sigma \cup \{(x : C) \geq n, (x : D) < n\}$, for all $n \in (0, 1]$ is not satisfiable. If $\Sigma \models C \sqsubseteq D$ then for all models \mathcal{I} of Σ , $\forall x \in \Delta^{\mathcal{I}}. C^{\mathcal{I}}(x) \leq D^{\mathcal{I}}(x)$, so Σ' is unsatisfiable, otherwise there would exist some \mathcal{I} with $w \in \Delta^{\mathcal{I}}$ and $n' \in (0, 1]$ s.t. $C^{\mathcal{I}}(w) = n' \geq n' > D^{\mathcal{I}}(w)$, which leads to absurd. For the converse suppose that \mathcal{I} is a model of Σ , Σ' is unsatisfiable, but to the contrary $C \not\sqsubseteq D$. Then there exists some $w \in \Delta^{\mathcal{I}}$ s.t. $C^{\mathcal{I}}(w) > D^{\mathcal{I}}(w)$. Extending \mathcal{I} to \mathcal{I}' s.t. $x^{\mathcal{I}'} = w$ and $C^{\mathcal{I}'}(x^{\mathcal{I}'}) = n \in (0, 1]$ we get a model of Σ and $\{(x : C) \geq n, (x : C) < n\}$. So \mathcal{I}' is a model of Σ' .
- $\Sigma \models \text{Trans}(R)$ iff $\Sigma' = \Sigma \cup \{(x : \exists R. (\exists R. \{y\})) \geq n, (x : \exists R. \{y\}) < n\}$ is not satisfiable, for all $n \in (0, 1]$. Suppose that $\Sigma \models \text{Trans}(R)$. Then in every model \mathcal{I} of Σ it holds, $\forall x, y \in \Delta^{\mathcal{I}}. R^{\mathcal{I}}(x, y) \geq \sup_{z \in \Delta^{\mathcal{I}}} t(R^{\mathcal{I}}(x, z), R^{\mathcal{I}}(z, y))$. Based on the boundary condition of the t-norm operation this inequation can be rewritten as,

$$\forall x, y \in \Delta^{\mathcal{I}}. t(R^{\mathcal{I}}(x, y), 1) \geq \sup_{z \in \Delta^{\mathcal{I}}} t(R^{\mathcal{I}}(x, z), t(R^{\mathcal{I}}(z, y), 1)).$$

Using the semantics of nominals we finally obtain

$$\forall x, y \in \Delta^{\mathcal{I}}. t(R^{\mathcal{I}}(x, y), \{y\}^{\mathcal{I}}(y)) \geq \sup_{z \in \Delta^{\mathcal{I}}} t(R^{\mathcal{I}}(x, z), t(R^{\mathcal{I}}(z, y), \{y\}^{\mathcal{I}}(y))).$$

The t-norm operation of the right-hand side is also the supremum of the t-norm operations, $t(R^{\mathcal{I}}(z, w), \{y\}^{\mathcal{I}}(w))$, for all $w \in \Delta^{\mathcal{I}}$, because if $w \neq y$ then due to the semantics of nominals we would have $\{y\}^{\mathcal{I}}(w) = 0$ and thus $t(a, 0) = 0$. This supremum is obviously an existential restriction on z . Similarly we get an existential restriction for x in the left side as well as in the right side of the inequation. Thus, we get $\forall x, y \in \Delta^{\mathcal{I}}. (\exists R. \{y\})^{\mathcal{I}}(x) \geq (\exists R. (\exists R. \{y\}))^{\mathcal{I}}(x)$. Hence, every model \mathcal{I} of Σ does not satisfy $\{(x : \exists R. (\exists R. \{y\})) \geq n, (x : \exists R. \{y\}) < n\}$ for any $n \in (0, 1]$.

For the converse suppose that \mathcal{I} is a model of Σ , Σ' is unsatisfiable but to the contrary there are some $a, c \in \Delta^{\mathcal{I}}$ for which $R^{\mathcal{I}}(a, c) < \sup_b t(R^{\mathcal{I}}(a, b), R^{\mathcal{I}}(b, c))$. Working in

a similar way as above we get $(\exists R.\{c\})^{\mathcal{I}}(a) < (\exists R.(\exists R.\{c\}))^{\mathcal{I}}(a)$. Extending \mathcal{I} to \mathcal{I}' such that $x^{\mathcal{I}'} = a$ and $y^{\mathcal{I}'} = c$ and $(\exists R.(\exists R.\{c\}))^{\mathcal{I}'}(a) = n$, for some $n \in (0, 1]$, we can devise an interpretation that satisfies Σ' , for some $n \in (0, 1]$, which is absurd.

- $\Sigma \models R \sqsubseteq S$ iff $\Sigma' = \Sigma \cup \{(x : \exists R.\{y\}) \geq n, (x : \exists S.\{y\}) < n\}$ is unsatisfiable, for all $n \in (0, 1]$. Suppose that $\Sigma \models R \sqsubseteq S$. Then in every model \mathcal{I} of Σ we have that $\forall x, y \in \Delta^{\mathcal{I}}. R^{\mathcal{I}}(x, y) \leq S^{\mathcal{I}}(x, y)$. Working in a similar way as in the previous case we have the following deduction steps:

$$\begin{aligned}
\forall x, y \in \Delta^{\mathcal{I}}. R^{\mathcal{I}}(x, y) &\leq S^{\mathcal{I}}(x, y) && \Rightarrow \\
\forall x, y \in \Delta^{\mathcal{I}}. t(R^{\mathcal{I}}(x, y), 1) &\leq t(S^{\mathcal{I}}(x, y), 1) && \Rightarrow \\
t(R^{\mathcal{I}}(x, y), \{y\}^{\mathcal{I}}(y)) &\leq t(S^{\mathcal{I}}(x, y), \{y\}^{\mathcal{I}}(y)) && \Rightarrow \\
\sup_{z \in \Delta^{\mathcal{I}}} t(R^{\mathcal{I}}(x, z), \{y\}^{\mathcal{I}}(z)) &\leq \sup_{z \in \Delta^{\mathcal{I}}} t(S^{\mathcal{I}}(x, z), \{y\}^{\mathcal{I}}(z)) && \Rightarrow \\
(\exists R.\{y\})^{\mathcal{I}}(x) &\leq (\exists S.\{y\})^{\mathcal{I}}(x).
\end{aligned}$$

Thus, there is no interpretation \mathcal{I}' that satisfies, Σ' .

For the converse, suppose that \mathcal{I} is a model of Σ , Σ' is unsatisfiable, but to the contrary $R \not\sqsubseteq S$. This means that there exist $a, b \in \Delta^{\mathcal{I}}$ s.t. $R^{\mathcal{I}}(a, b) > S^{\mathcal{I}}(a, b)$. Working in a similar way we can obtain $(\exists R.\{b\})^{\mathcal{I}}(a) > (\exists S.\{b\})^{\mathcal{I}}(a)$. Now extending \mathcal{I} to \mathcal{I}' such that $x^{\mathcal{I}'} = a$, $y^{\mathcal{I}'} = b$ and $(\exists R.\{b\})^{\mathcal{I}'}(x^{\mathcal{I}'}) = n$, for some $n \in (0, 1]$ we get $(\exists R.\{y\})^{\mathcal{I}'}(x^{\mathcal{I}'}) \geq n > (\exists S.\{y\})^{\mathcal{I}'}(x^{\mathcal{I}'})$, which is a model of Σ' , leading to absurd. \square

Proof of Theorem 5.3. We will work in a similar way as in the previous proof.

- $\Sigma \models R \sqsubseteq S$, iff $\Sigma' = \Sigma \cup \{(x : \exists R.B) \geq n, (x : \exists S.B) < n\}$, where B is a new concept not present in Σ . In every model \mathcal{I} of Σ we have that $\forall a, b \in \Delta^{\mathcal{I}}. R^{\mathcal{I}}(a, b) \leq S^{\mathcal{I}}(a, b)$. By using the monotonicity property of t-norms we get $\forall a, b \in \Delta^{\mathcal{I}}. t(R^{\mathcal{I}}(a, b), B^{\mathcal{I}}(b)) \leq t(S^{\mathcal{I}}(a, b), B^{\mathcal{I}}(b))$ and since this holds for all $b \in \Delta^{\mathcal{I}}$ it would also hold for the supremum, thus

$$\begin{aligned}
\forall a, b \in \Delta^{\mathcal{I}}. \sup_b t(R^{\mathcal{I}}(a, b), B^{\mathcal{I}}(b)) &\leq \sup_b t(S^{\mathcal{I}}(a, b), B^{\mathcal{I}}(b)) && \Rightarrow \\
\forall a, b \in \Delta^{\mathcal{I}}. (\exists R.B)^{\mathcal{I}}(a) &\leq (\exists S.B)^{\mathcal{I}}(a).
\end{aligned}$$

Since this holds for all \mathcal{I} , $\{(x : \exists R.B) \geq n, (x : \exists S.B) < n\}$ is unsatisfiable. For the converse case suppose that \mathcal{I} is a model of Σ , Σ' is unsatisfiable, but to the contrary there are some $a, b \in \Delta^{\mathcal{I}}$, $R^{\mathcal{I}}(a, b) = p_1 > p_2 = S^{\mathcal{I}}(a, b)$. We have that $R^{\mathcal{I}}(a, b) = t(R^{\mathcal{I}}(a, b), 1)$. Extend \mathcal{I} to \mathcal{I}' such that $B^{\mathcal{I}'}(b) = 1$ and $B^{\mathcal{I}'}(c) = 0$, for all $c \neq b$. Observe that the reason why we can perform such an extension is because B does not appear anywhere in the KB so its membership function $B^{\mathcal{I}'}$ is not restricted by any axiom in Σ . Then we have $t(R^{\mathcal{I}'}(a, b), 1) = t(R^{\mathcal{I}'}(a, b), B^{\mathcal{I}'}(b)) =$

$\sup_{c \in \Delta^{\mathcal{I}}} t(R^{\mathcal{I}}(a, c), B^{\mathcal{I}}(c)) = (\exists R.B)^{\mathcal{I}}(a)$. Similarly, $p_2 = (\exists S.B)^{\mathcal{I}}(x)$. Then we can extend \mathcal{I} to \mathcal{I}' , such that $x^{\mathcal{I}'} = a$, $y^{\mathcal{I}'} = b$ and $n = (\exists R.B)^{\mathcal{I}}(x^{\mathcal{I}'})$, thus constructing an interpretation which is a model of Σ' for $n = p_1$.

- $\Sigma \models \text{Trans}(R)$ iff $\Sigma' = \Sigma \cup \{(x : \exists R.(\exists R.B)) \geq n, (x : \exists R.B) < n\}$ is unsatisfiable, for all $n \in (0, 1]$. Suppose that $\Sigma \models \text{Trans}(R)$. Then in every model \mathcal{I} of Σ it holds that, $\forall x, y \in \Delta^{\mathcal{I}}. R^{\mathcal{I}}(x, y) \geq \sup_z t(R^{\mathcal{I}}(x, z), R^{\mathcal{I}}(z, y))$. Based on the monotonicity property of t-norms we get, $t(R^{\mathcal{I}}(x, y), B^{\mathcal{I}}(y)) \geq \sup_z t(R^{\mathcal{I}}(x, z), t(R^{\mathcal{I}}(z, y), B^{\mathcal{I}}(y)))$. Since this holds for all $x, y \in \Delta^{\mathcal{I}}$ it would also hold for the supremum of y , hence, $\sup_y t(R^{\mathcal{I}}(x, y), B^{\mathcal{I}}(y)) \geq \sup_z t(R^{\mathcal{I}}(x, z), \sup_y t(R^{\mathcal{I}}(z, y), B^{\mathcal{I}}(y)))$, which can be finally written as

$$(\exists R.B)^{\mathcal{I}}(x) \geq \sup_z t(R^{\mathcal{I}}(x, z), (\exists R.B)^{\mathcal{I}}(z)) = (\exists R.(\exists R.B))^{\mathcal{I}}(x).$$

Since this holds for all models \mathcal{I} , Σ' is unsatisfiable. For the converse case suppose that \mathcal{I} is a model of Σ , Σ' is unsatisfiable, but to the contrary there are some $a, b \in \Delta^{\mathcal{I}}$, such that

$$\sup_w t(R^{\mathcal{I}}(a, w), R^{\mathcal{I}}(w, b)) = p_1 > p_2 = R^{\mathcal{I}}(a, b).$$

We have that $R^{\mathcal{I}}(a, b) = t(R^{\mathcal{I}}(a, b), 1)$. Extend \mathcal{I} to \mathcal{I}' such that $B^{\mathcal{I}'}(b) = 1$ and $B^{\mathcal{I}'}(w) = 0$, for all $w \neq b$. Then we have $t(R^{\mathcal{I}'}(a, w), B^{\mathcal{I}'}(w)) = t(R^{\mathcal{I}'}(a, w), 0) = 0$. Then we have $t(R^{\mathcal{I}'}(a, w), B^{\mathcal{I}'}(w)) = (\exists R.B)^{\mathcal{I}'}(a)$. Similarly, $p_2 = (\exists R.(\exists R.B))^{\mathcal{I}'}(a)$. Then we can extend \mathcal{I} to \mathcal{I}' , such that $x^{\mathcal{I}'} = a$, $y^{\mathcal{I}'} = b$ and $n = (\exists R.B)^{\mathcal{I}'}(x^{\mathcal{I}'})$, thus constructing an interpretation which is a model of Σ' for $n = p_1$. \square

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